1. (a) Homomorphism. \( \text{Ker}(\varphi) = n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\} \). Onto but not 1-1.

(b) Not a homomorphism unless \( G \) is abelian. This is because \( \varphi(ab) = (ab)^{-1} = b^{-1}a^{-1} \) which may not be equal to \( \varphi(a)\varphi(b) = a^{-1}b^{-1} \).

(c) Homomorphism. \( \text{Ker}(\varphi) = \{e\} \). Onto and 1-1 (every element has a unique inverse).

(d) Homomorphism. \( \text{Ker}(\varphi) = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} \). Onto but not 1-1.

(e) Homomorphism. \( \text{Ker}(\varphi) = \{x \in G \mid x^n = e\} \) (elements in \( G \) whose order divides \( n\)). If \( (n, |G|) = 1 \), then \( \varphi \) is onto and 1-1. Otherwise \( \varphi \) is neither onto nor 1-1.

2. If \( x, y \in \varphi(G) \), then we can find \( x_g, y_g \in G \) such that \( \varphi(x_g) = x \) and \( \varphi(y_g) = y \). Thus \( xy = \varphi(x_g)\varphi(y_g) = \varphi(x_gy_g) \). Since \( G \) is a group, we have \( x_gy_g \in G \) and hence \( xy = \varphi(x_gy_g) \in \varphi(G) \). Similarly, we have \( x^{-1} = (\varphi(x_g))^{-1} = \varphi(x_g^{-1}) \in \varphi(G) \) since \( x_g^{-1} \in G \). Finally, note that \( e = \varphi(e) \in \varphi(G) \).

3. Suppose \( \varphi \) is an monomorphism. Then \( \varphi(x) = \varphi(y) \Rightarrow x = y \). If \( x \in \text{Ker}(\varphi) \), we have \( \varphi(x) = \varphi(e) = e \) which implies \( x = e \). Hence \( \text{Ker}(\varphi) \subset \{e\} \) and since \( e \in \text{Ker}(\varphi) \), we have \( \text{Ker}(\varphi) = \{e\} \).

Suppose \( \text{Ker}(\varphi) = \{e\} \). Let \( x, y \in G \) be such that \( \varphi(x) = \varphi(y) \). We have \( \varphi(x)\varphi(y)^{-1} = e \) which implies \( \varphi(xy^{-1}) = e \). Since \( \text{Ker}(\varphi) = \{e\} \), we have \( xy^{-1} = e \) which is equivalent to \( x = y \). Hence \( \varphi \) is injective.

4. Let \( x', y' \in G' \). Since \( \varphi \) is surjective, we can find \( x, y \in G \) such that \( \varphi(x) = x' \) and \( \varphi(y) = y' \). Since \( G \) is abelian, we have \( xy =yx \). Then

\[
x'y' = \varphi(x)\varphi(y) = \varphi(xy) = \varphi(yx) = \varphi(y)\varphi(x) = y'x',
\]

and hence \( G' \) is abelian.

5. If \( (x_1, y_1), (x_2, y_2), (x_3, y_3) \in G_1 \times G_2 \), then

\[
\left( (x_1, y_1)(x_2, y_2) \right)(x_3, y_3) = (x_1x_2, y_1y_2)(x_3, y_3) = (x_1x_2x_3, (y_1y_2)y_3) = (x_1(x_2x_3), y_1(y_2y_3)) = (x_1, y_1)(x_2x_3, y_2y_3) = (x_1, y_1)((x_2, y_2)(x_3, y_3)),
\]

so multiplication in \( G_1 \times G_2 \) is associative. The identity is \( (e, f) \) where \( e \) is the identity in \( G_1 \) and \( f \) is identity in \( G_2 \) since \( (e, f)(x, y) = (ex, fy) = (x, y) \) and \( (x, y)(e, f) = (x, y) \).
(xe, yf) = (x, y) for any (x, y) ∈ G₁ × G₂. If (x, y) ∈ G₁ × G₂, then

\[(x^{-1}, y^{-1})(x, y) = (x^{-1}x, y^{-1}y) = (e, f) = (xx^{-1}, yy^{-1}) = (x, y)(x^{-1}, y^{-1}),\]

so \((x, y)^{-1} = (x^{-1}, y^{-1})\). Hence we conclude that \(G₁ × G₂\) is a group under \(*\).

Now define \(ϕ : G₁ × G₂ → G₂ × G₁\) via \(ϕ(x, y) = (y, x)\). This is a homomorphism, since

\[
ϕ((x₁, y₁)(x₂, y₂)) = ϕ((x₁x₂, y₁y₂))
\]

\[
= (y₁y₂, x₁x₂)
\]

\[
= (y₁, x₁)(y₂, x₂)
\]

\[
= ϕ((x₁, y₁))ϕ((x₂, y₂))
\]

for all \(x₁, x₂ ∈ G₁\) and \(y₁, y₂ ∈ G₂\). Each \((y, x) ∈ G₂ × G₁\) is the image of \((x, y)\) under \(ϕ\), so \(ϕ\) is surjective. Also if \((x, y) ∈ \text{Ker}(ϕ)\), then \(ϕ(x, y) = (y, x) = (f, e)\), so \((x, y) = (e, f)\) is the identity element of \(G₁ × G₂\), and hence \(\text{Ker}(ϕ) = \{(e, f)\}\). Therefore \(ϕ\) is injective by question 3. We conclude that \(ϕ\) is an isomorphism and hence \(G₁ × G₂ ≅ G₂ × G₁\).

6. Let \(g, h ∈ G\). We need to prove that \(ψ(gh) = σ_{gh}\) is equal to \(ψ(g) ∘ ψ(h) = σ_gh ∘ σ_h\). If \(x ∈ G\) then \(σ_{gh}(x) = (gh)x(gh)^{-1}\) and

\[
(σ_gh ∘ σ_h)(x) = σ_gh(hxh^{-1}) = g(hxh^{-1})g^{-1} = (gh)x(gh)^{-1}.
\]

Hence we conclude that \(ψ(gh) = σ_{gh} = σ_gh ∘ σ_h = ψ(g) ∘ ψ(h)\), and \(ψ\) is a homomorphism.

Now let \(g ∈ \text{Ker}(ψ)\). We have \(ψ(g) = σ_gh = i_G\), the identity map on \(G\). Hence for every \(x ∈ G\) we have \(σ_gh(x) = gxg^{-1} = x\), which implies \(gx = xg\). Therefore \(g ∈ Z(G)\) and hence \(\text{Ker}(ψ) ⊂ Z(G)\). Conversely, let \(g ∈ Z(G)\). We have \(gx = xg\) for all \(x ∈ G\). Hence \(gxg^{-1} = σ_gh(x) = x = i_G(x)\) for every \(x ∈ X\). Therefore \(g ∈ \text{Ker}(ψ)\). This shows that \(Z(G) ⊂ \text{Ker}(ψ)\) and hence \(\text{Ker}(ψ) = Z(G)\).

7. Let \(n ∈ θ(N)\) and \(g ∈ G\). We need to show that \(g^{-1}ng ∈ θ(N)\). For this, set \(n' := θ^{-1}(n)\) and \(g' := θ^{-1}(g)\), and note that \(g'^{-1}n'g' ∈ N\) (since \(n' ∈ N\) and \(N ⊆ G\)). It follows that

\[
g^{-1}ng = θ(g'^{-1}θ(n')θ(g')) = θ(g'^{-1}n'g') ∈ θ(N).
\]

8. Pick a point \(v₁ ∈ \mathbb{R}²\) fixed by \(s\) (for example \(v₁ := (0, 1)\)). Also set \(v₂ := r(v₁)\) and \(v₃ := r(v₂)\). The idea is that \(V := \{v₁, v₂, v₃\}\) is the set of vertices of a triangle in the plane. Since \(r\) has order 3, \(r(v₃) = v₁\) and hence \(r(V) = V\). Moreover

\[
s(v₂) = s(r(v₁)) = (sr)(v₁) = (r^{-1}s)(v₁) = r^{-1}(v₁) = v₃
\]
and similarly \( s(v_3) = v_2 \). It follows that \( s(V) = V \), and therefore \( g(V) = V \) for all \( g \in D_6 \).

So if \( g \in D_6 \), we can define its restriction \( r(g) : V \to V \) (which just sends \( v \mapsto g(v) \)). Note that \( r(gh) = r(g)r(h) \) for all \( g, h \in D_6 \); indeed

\[
(r(gh))(v) = (gh)(v) = g(h(v)) = g(r(h)(v)) = r(g)(r(h)(v)) = (r(g)r(h))(v)
\]

for all \( v \in V \). In particular \( r(g^{-1})r(g) = r(g^{-1}g) = r(e) = e \) and similarly \( r(g)r(g^{-1}) = e \); in other words \( r(g^{-1}) \) is the inverse of \( r(g) \). Therefore \( r \) defines a function \( D_6 \to A(V) \), which is a homomorphism by the above calculation.

To show that \( r \) is an isomorphism, it suffices to prove injectivity, because \( D_6 \) and \( A(V) \) are finite sets of the same size. If \( g \in \text{Ker}(r) \) then the restriction of \( g \) to \( V \) is the identity, so \( g \) fixes \( v_1 \) and \( v_2 \). You can check that these two vectors give a basis for \( \mathbb{R}^2 \). Since the elements of \( D_6 \) are linear transformations (because \( r \) and \( s \) are), it follows that \( g \) fixes every vector in \( \mathbb{R}^2 \). This shows that \( \text{Ker}(r) = \{ e \} \), as required.

It remains to prove that \( A(V) \cong S_3 \). For this, define \( f : \{1, 2, 3\} \to V \) by \( f(i) = v_i \). I hope it is clear that \( f \) is a bijection. Next, we can define \( \varphi : S_3 \to A(V) \) by \( \varphi(x) = f^{-1}xf \). If \( x, y \in S_3 \) then

\[
\varphi(xy) = f^{-1}xyf = f^{-1}xff^{-1}xf = \varphi(x)\varphi(y),
\]

so \( \varphi \) is a homomorphism. It is bijective because it has an inverse given by \( g \mapsto fgf^{-1} \). Therefore \( \varphi \) is an isomorphism, so \( D_6 \cong A(V) \cong S_3 \).

For the challenge problem, let \( G \) be a nonabelian group of order 6. By Lagrange’s theorem, elements of \( G \) can have orders 1, 2, 3 and 6. If \( g \in G \) has order 6, then \( \langle g \rangle \) is a subgroup of \( G \) with order 6, so \( \langle g \rangle = G \), contradicting the assumption that \( G \) is nonabelian. Therefore \( G \) only has elements of orders 1, 2 and 3. The set \( \{ g \in G \mid o(g) = 3 \} \) can be partitioned into subsets of the form \( \{ g, g^{-1} \} \) (with \( g \neq g^{-1} \)) so \( G \) has an even number of elements of order 3, i.e. either 0, 2, 4 or 6. It cannot have 6 because \( o(e) = 1 \), and it cannot have 0 by Problem 2 on Homework 2 (if \( g^2 = e \) for all \( g \in G \), then \( G \) is abelian).

Next, let \( g, h \in G \) have orders 2 and 3 respectively. Suppose for a contradiction that \( gh = hg \). Since \( C(g) \) contains \( g \) and \( h \), both 2 and 3 divide \( |C(g)| \) by Lagrange’s theorem. Therefore \( C(g) = G \), and similarly \( C(h) = G \). Given any \( g' \in G \) of order 2, it follows that \( g'h = hg' \), and the same argument shows that \( C(g') = G \). Similarly \( C(h') = G \) for any \( h' \in G \) of order 3. Therefore every element of \( G \) is central, which is impossible because \( G \) is nonabelian. The upshot is that \( gh \neq hg \), or equivalently \( h^{-1}gh \neq g \). This shows that \( G \) has at least two elements of order 2 (namely \( g \) and \( h^{-1}gh \)), so it cannot have 4 of order 3. Therefore \( G \) has 3 elements of order 2 and 2 of order 3.
Let $T := \{g \in G \mid \phi(g) = 2\}$. As above we can easily show that $A(T) \cong S_3$. Each $g \in G$ defines a function $\phi(g) : T \to T$ by sending $t \mapsto g t g^{-1}$. If $g, h \in G$ then

\[ \phi(gh)(t) = g h t (g h)^{-1} = g h t h^{-1} g^{-1} = g \phi(h)(t) g^{-1} = \phi(g)(\phi(h)(t)) = (\phi(g) \phi(h))(t) \]

for all $t \in T$, so $\phi(gh) = \phi(g) \phi(h)$. It follows that $\phi(g^{-1})$ is the inverse of $\phi(g)$, and in particular $\phi$ defines a function $G \to A(T)$. The above calculation shows that $\phi$ is a homomorphism. As above it remains to show that $\phi$ is injective. For this, let $g \in \text{Ker}(\phi)$. Since $\phi(g)$ is the identity function $T \to T$, $g$ commutes with every element of $T$. In other words $T \subseteq C(g)$, which implies that $|C(g)| \geq 4$ (since $e \in C(g)$ also). Therefore $g$ is central by Lagrange’s theorem. This forces $g = e$, because the elements of order 2 do not commute with those of order 3, and vice versa.

9 & 10. We will tackle these at the same time, because if $\text{Isom}(T) \cong S_4$ then $|\text{Isom}(T)| = 24$. The $3 \times 3$ identity matrix obviously belongs to $\text{Isom}(T)$. Moreover, if $B, C \in \text{Isom}(T)$ then $B$ and $C$ give bijections $T \to T$, so $BC$ and $B^{-1}$ also give bijections $T \to T$, and hence $BC, B^{-1} \in \text{Isom}(T)$. Thus $\text{Isom}(T) \leq \text{GL}_3(\mathbb{R})$, and in particular $\text{Isom}(T)$ is a group.

Let $V = \{v_1, v_2, v_3, v_4\}$ be the set of vertices of $T$. The second paragraph of Problem 8 (with some minor adjustments) shows that the restriction map $r : \text{Isom}(T) \to A(V)$ is a homomorphism. It is injective by another similar argument: any three vertices give a basis for $\mathbb{R}^3$, and the elements of $\text{Isom}(T)$ are literally matrices this time. To check that any three vertices give a basis, it might be easiest to check the first three directly, then express all four in that basis. Since $v_1 + v_2 + v_3 + v_4 = 0$ you will get $\{e_1, e_2, e_3, (-1, -1, -1)\}$, where $e_i$ is the $i$th standard basis vector (the $i$th column of the $3 \times 3$ identity matrix). This set is a bit easier to deal with.

To prove that $r$ is onto, we have to do some work (because $|\text{Isom}(T)|$ is unknown). For each $f \in A(V)$ let $M_f$ be the $3 \times 3$ matrix with columns $f(v_1), f(v_2)$ and $f(v_3)$. Since any three vertices give a basis for $\mathbb{R}^3$, these matrices are all invertible. By definition $M_f e_i = f(v_i)$, and hence $M_f M_e^{-1} v_i = f(v_i)$, for each $i \in \{1, 2, 3\}$. It follows that

\[ M_f M_e^{-1} v_4 = M_f M_e^{-1}(-v_1 - v_2 - v_3) = -f(v_1) - f(v_2) - f(v_3) = f(v_4) \]

(the last step follows from the formula $v_1 + v_2 + v_3 + v_4 = 0$ and the fact that $f$ is bijective). This shows that $r(M_f M_e^{-1}) = f$, if you are willing to accept that $M_f M_e^{-1} \in \text{Isom}(T)$. To prove this, note that $T$ is the convex hull of $V$, which means it is the set of convex linear combinations of elements of $V$. A linear combination $\sum_{i=1}^4 a_i v_i$ is convex provided that the coefficients are nonnegative and add up to 1. So if $t \in T$ then $t = \sum_{i=1}^4 a_i v_i$ for some $a_i \geq 0$ with $\sum_{i=1}^4 a_i = 1$, and hence $M_f M_e^{-1} t = \sum_{i=1}^4 a_i M_f M_e^{-1} v_i = \sum_{i=1}^4 a_i f(v_i) \in T$. 

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