Lecture 1: Graphs and their properties

Chapters 1.1, 1.3, 1.4

Topics for today:

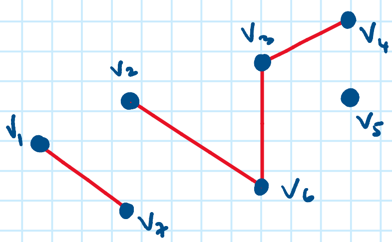
- What is a graph?
- Terminology and basic properties
- Degrees and neighborhoods
- Special types of graphs

What is a graph?

- A pair of sets (V, E)
  - V contains **nodes** or **vertices** (singular **vertex**)
  - E contains **edges** or **unordered** pairs of vertices

SIMPLE GRAPHS

- Example:



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \qquad E = \{(v_1,v_7), (v_2,v_6), (v_3,v_6), (v_4,v_3)\}$$

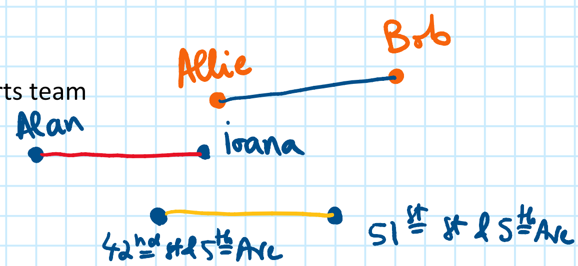
Which relationships can be modeled by a graph?

V E
- Group of people, people sharing a favorite sports team

V E
- Mathematicians, co-authors (of papers)

V E
- Intersections, roads (in a city)

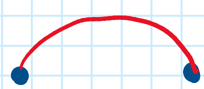
V E
- Book, words



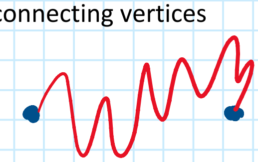
Drawing graphs:

- Vertices are points or nodes
- Edges are
    - Segments connecting vertices

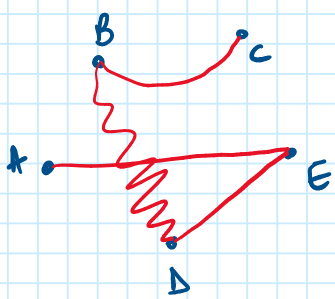
    - Arcs connecting vertices



    - Curves connecting vertices



    - **Example**:



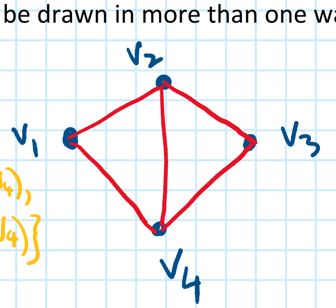
$\underline{V} = \{A, B, C, D, E\}$

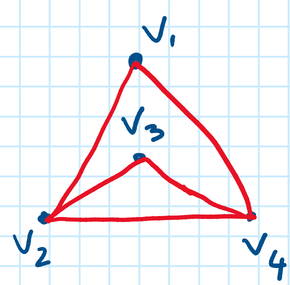
$\underline{E} = \{AE, BC, BD, DE\}$

- Graphs can be drawn in more than one way

$V = \{v_1, v_2, v_3, v_4\}$

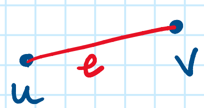
$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_2, v_4)\}$



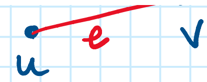


Adjacency and incidence:

- (u,v) edge then u and v are **adjacent**
- e = (u,v) we say u and v are **endpoints** of e



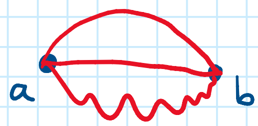
- e = (u,v)   we say u and v are **endpoints** of e



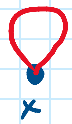
- e is a edge **incident** to v if v is an endpoint for e

Other types of graphs:

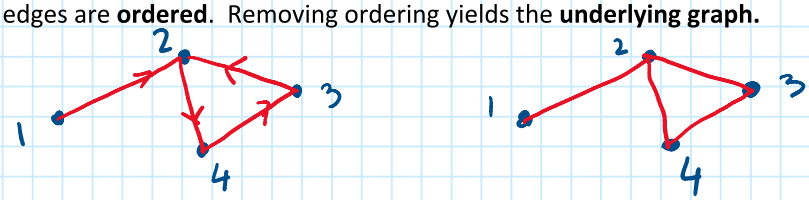
- **Multigraphs**, allow for **multiple edges** between same pair of vertices



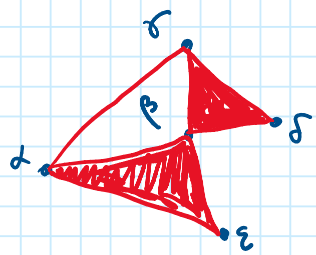
- **Pseudographs**, allow for loops



- **Digraphs**, edges are **ordered**. Removing ordering yields the **underlying graph.**

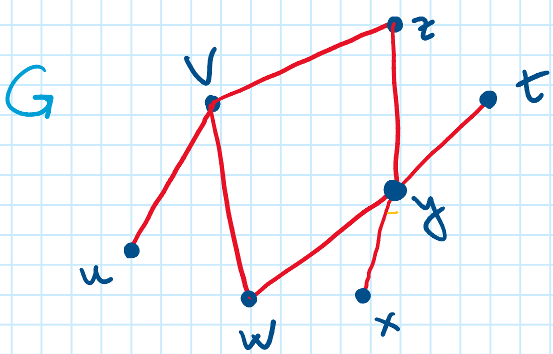


- **Hypergraphs**, edges are sets of **more** than 2 vertices



Degrees and neighborhoods

- **v** a vertex, the **neighborhood of v**, $N_G(v)$, is the **set of vertices adjacent to v**



$N_G(v) = \{u, w, z\}$ $N_G(y) = \{w, x, t, z\}$

$d_G(v) = |N_G(v)| = 3$ $d_G(y) = |N_G(y)| = 4$

$\delta(G) = 1$ $\Delta(G) = 4$

- **Degree of v**, denoted $d_G(v)$, is the size of the neighborhood of v, $d_G(v) = |N_G(v)|$

- If $d_G(v) = 0$, we say **v** is **isolated**

- We may drop G if it is clear from the context

$N(v) = N_G(v)$, $d_G(v) = d(v)$

- **Minimum** degree of a graph, **maximum** degree of a graph

$$\boldsymbol{\delta(G) = \min\{d_G(v) : v \in V\}}$$

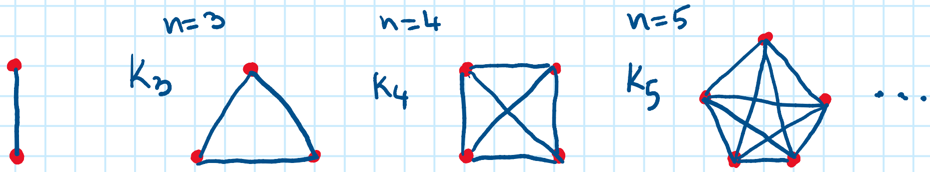
$$\boldsymbol{\Delta(G) = \max\{d_G(v) : v \in V\}}$$

Special types of graphs:

- **Complete graphs** or **cliques** K_n $n = \#$ of vertices $= |V|$

    - Edge set is all possible edges

n=2 K_2 n=3 K_3 n=4 K_4 n=5 K_5



    - Total number of edges: $\binom{n}{2} = \frac{n(n-1)}{2}$

- **Bipartite** graphs

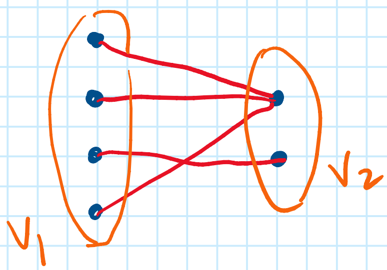
    - $V = V_1 \cup V_2$ is a partition of V

    - Edges occur only between V_1 and V_2

V_1, V_2 are "classes"

$|V_1| = m$, $|V_2| = n$

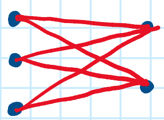
m=4
n=2



    - **Complete** bipartite graph $K_{m,n}$

$K_{3,2}$

m=3
n=2



E consists of all possible edges between

V_1 , $|V_1| = m$

and

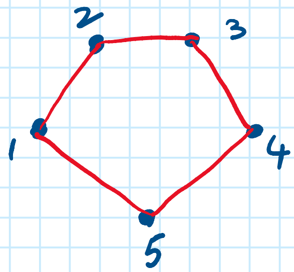
V_2 , $|V_2| = n$

Total number of edges $= m \cdot n$

- A **k-cycle** C_k with $V = \{1,2,3,\dots,k\}$ has $E = \{(1,2),(2,3),\dots,(k-1,k),(k,1)\}$

$k = 5$

length = # of edges

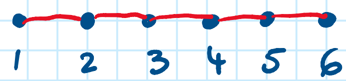


C_5 has length 5

- A **k-path** P_k with $V = \{1,2,3,\dots,k+1\}$ has $E = \{(1,2),(2,3),\dots,(k,k+1)\}$

$k+1 = 6$

length = # of edges



P_5

P_5 has length 5