Lecture 2: Digraphs, subgraphs, the Handshake Lemma

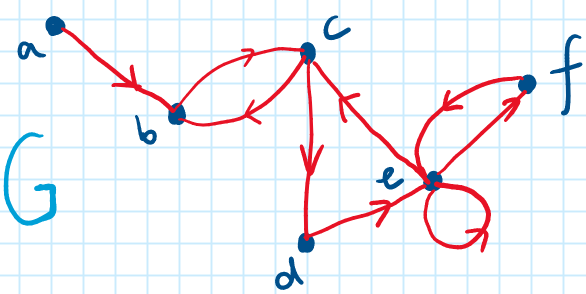
Covers chapters 1.5, 1.6, 1.7

Topics for today:

- Digraphs; what do neighborhoods and degrees look like?
- Subgraphs, induced subgraphs, removing edges and vertices
- A simple but powerful ("Handshake") Lemma

Digraphs (directed graphs, networks)

- (V, E)
- V vertices, E **arcs** (**oriented** pairs of vertices, **loops**, many arcs between two vertices)



$N^+(e) = \{c, f, e\}$

$N^-(e) = \{d, f, e\}$

$d^+(e) = 3$

$d^-(e) = 3$

- Every vertex v has an in-neighborhood and an out-neighborhood:

$$N^{\text{out}}(v) = N^+(v) = \{u : (v, u) \in E\}$$

$$N^{\text{in}}(v) = N^-(v) = \{u : (u, v) \in E\}$$

- Consequently, the in-degrees and out-degrees of digraphs are given as

$$\mathrm{d}^{\text{out}}(v) = d^+(v) = |N^+(v)|$$

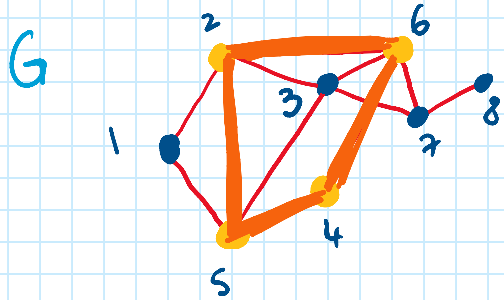
$$\mathrm{d}^{\text{in}}(v) = d^-(v) = |N^-(v)|$$

Subgraphs

- A graph $H = (V_1, E_1)$ is a **subgraph** of the graph $G = (V, E)$ if

- $V_1 \subseteq V$ and $E_1 \subseteq E$
- In addition, if $V_1 = V$, then **H** is a **spanning subgraph** of **G**

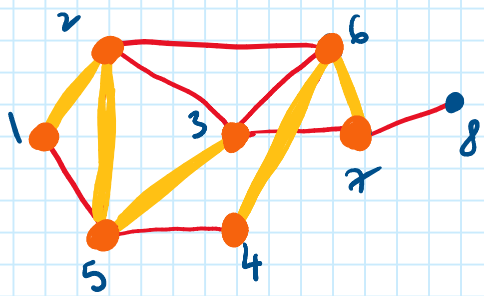
- A subgraph **H** of **G** is **induced by** a subset of vertices **X** if E(H) consists of all edges in **G** between vertices in **X**



$X = \{2, 4, 5, 6\}$

$E(H) = \{(2,5), (5,4), (4,6), (2,6)\}$

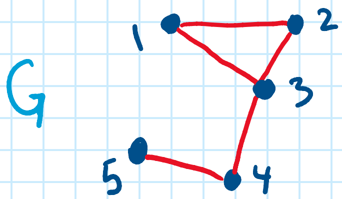
- A subgraph H of G is **induced** by a subset of edges $E(H) \subseteq E$ if V(H) consists of all the vertices in V which are **incident** to edges in $E(H)$.



$E(H) = \{(1,2), (2,5), (5,3), (6,4), (7,6)\}$

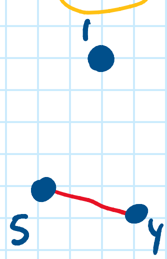
$X = \{1, 2, 3, 4, 5, 6, 7\}$

- When we want to "remove" a set of vertices X, we obtain the subgraph induced by $V \setminus X$. We denote the resulting graph by **G - X.**

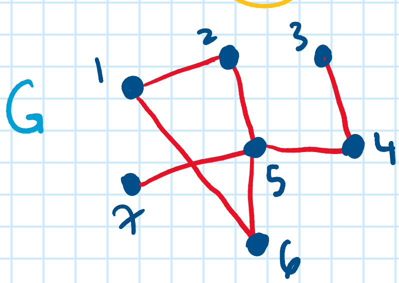


$X = \{2, 3\}$

$G - X$

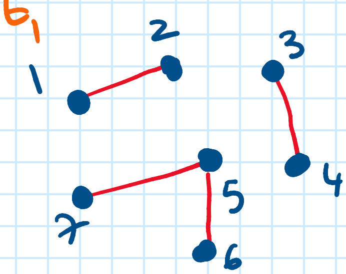


- When we want to "remove" a set of edges E_1 , we take the subgraph that results when each of those edges is removed. We denote the resulting subgraph by G - E_1 .



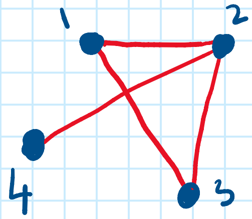
$E_1 = \{(5,4), (1,6), (2,5)\}$

$G - E_1$



The Handshake Lemma (also known as the "handshaking" Lemma)

G



$d(1) = 2$

$d(2) = 3$ $\qquad |E| = 4$

$d(3) = 2$

$d(4) = 1$

$d(1) + d(2) + d(3) + d(4) = 8 = 2|E|$

Lemma. For any simple graph $G = (V, E)$,

$$\sum_{v \in V} d_G(v) = 2|E|$$

Proof. A very useful technique in this course, based on counting something in two different ways.

We will count the number of (distinct) pairs (e, v) such that $e \in E$, $v \in V$, and v is incident to e.

count #1. Based on edges. Let $e \in E$.

How many pairs involve e?

How many vertices can form a pair with e?

2 vertices: $e = (u, v)$

2 pairs will be counted: (e, u), (e, v).

Every edge appears in 2 pairs.

TOTAL # of pairs: $\boxed{2|E|}$

count # 2. Based on vertices. Let $v \in V$.

How many pairs involve v?

How many edges are incident to v?

As many as $|N_G(v)| = d_G(v)$

As many as $|N_G(v)| = d_G(v)$

Every vertex appears in $d_G(v)$ pairs.

TOTAL # of pairs: $\boxed{\sum_{v \in V} d_G(v)}$

Therefore $\sum_{v \in V} d_G(v) = 2|E|$.

Corollary. The number of vertices of odd degree in a graph is always even.

Proof. By contradiction. Suppose the number were odd. (for some graph G)

$$2|E| = \sum_{v \in V} d_G(v) = \underbrace{\sum_{\substack{v \in V \\ d_G(v) \text{ odd}}} d_G(v)}_{\substack{\text{Summing an odd} \\ \text{number of odd degrees} \\ = \text{odd number}}} + \underbrace{\sum_{\substack{v \in V \\ d_G(v) \text{ even}}} d_G(v)}_{\text{even number}}$$

odd number

But $2|E|$ is even! Contradiction!

Therefore the corollary is true.

Example. We will call a graph in which all degrees are equal to **r** a **r**-regular graph.
How many edges does a 3-regular graph on 5 vertices have?

Answer.

This is a TRICK QUESTION....

NO SUCH GRAPH EXISTS!

NO REGULAR GRAPH WITH ODD DEGREE

NO REGULAR GRAPH WITH ODD DEGREE
AND ODD NUMBER OF VERTICES EXISTS!