Conjectures for the delta operator expression $\Delta'_{ek} \Delta_{hr} e_n$

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Joint work with Andy Wilson

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Partition and Tableau

- \( \lambda = \lambda_1, \ldots, \lambda_k \) is a partition of \( n \) if \( \lambda_1 \geq \ldots \geq \lambda_k \) and \( \sum_{i=1}^{k} \lambda_i = n \), written \( \lambda \vdash n \).

- Ex. \( \lambda \vdash 3 : (3), (2, 1), (1, 1, 1) \).
Partition and Tableau

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- Ex. $\lambda \vdash 3 : (3), (2, 1), (1, 1, 1)$.

- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4, 2, 1) \vdash 7$ corresponds to \[
\begin{array}{ccccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
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  We can fill the cells of the Ferrers diagram with integers.

- Injective tableau: $\lambda \rightarrow \mathbb{Z}_+$,

  \[
  \begin{array}{cccc}
  4 & 1 & 5 \\
  2 & 6 & 2 & 4 \\
  \end{array}
  \]
Partition and Tableau

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- **Injective tableau:** \( \lambda \rightarrow \mathbb{Z}_+ \),

  \[
  \begin{array}{cccc}
  4 & 1 & 5 & 2 \\
  6 & 2 & 4 & \\
  \end{array}
  \]

- **Column strict tableau:**

  \[
  \begin{array}{cccc}
  5 & 2 & 3 & \\
  1 & 1 & 3 & 4 \\
  \end{array}
  \]
Symmetric Functions

\[ S_n = \{ \sigma : \sigma \text{ is a permutation of } [n] \} \text{ is the } n^{\text{th}} \text{ symmetric group.} \]
Symmetric Functions

- $S_n = \{\sigma : \sigma \text{ is a permutation of } [n]\}$ is the $n^{th}$ symmetric group.

- $f(X) \in \mathbb{R}[[x]]$ is a symmetric function if $f(X) = f(\sigma(X))$ for any permutation $\sigma$.

- Ex.
  \[f(x_1, x_2, x_3) = 3x_1x_2 + 3x_1x_3 + 3x_2x_3 + \cdots + 5x_1^2x_2 + 5x_1x_2^2 + 5x_1^2x_3 + \cdots\]
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- The ring of symmetric functions has several bases: $\{s_\lambda\}, \{e_\lambda\}, \ldots$.

- $e_n = \sum_{i_1 < \cdots < i_n} x_{i_1}x_{i_2} \cdots x_{i_n}$, and $e_\lambda = e_{\lambda_1}e_{\lambda_2} \cdots e_{\lambda_k}$. 
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- The ring of symmetric functions has several bases: \(\{s_\lambda\}, \{e_\lambda\}, \ldots\).

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- \[
  s_\lambda = \sum_{T \text{ a column strict tableau of shape } \lambda} X^T.
  \]
Quasi-symmetric Functions

- $f(X) \in \mathbb{R}[[x]]$ is a quasi-symmetric function if for each composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, the coefficient of the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$. 

F \quad S = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n, \quad i_j < i_{j+1}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_n}^{\alpha_n}$ is the fundamental quasi-symmetric function associated with a set $S \subset [n-1]$. 

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Quasi-symmetric Functions

- \( f(\mathbf{X}) \in \mathbb{R}[[\mathbf{x}]] \) is a **quasi-symmetric function** if for each composition \( \alpha = (\alpha_1, \ldots, \alpha_k) \), the coefficient of the monomial \( x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \) is equal to the coefficient of the monomial \( x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \) for any strictly increasing sequence of positive integers \( i_1 < i_2 < \cdots < i_k \).

- \( F_S = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n, i_j < i_{j+1} \text{ if } j \in S} x_{i_1} x_{i_2} \cdots x_{i_n} \) is the **fundamental quasi-symmetric function** associated with a set \( S \subset [n-1] \).
The Macdonald polynomial \( \tilde{H}_\mu(X; q, t) \) is a \( q, t \)-weighted symmetric function given by

\[
\tilde{H}_\mu(X; q, t) = \sum_{\sigma: \mu \to \mathbb{Z}_+ \text{ injective tableau}} q^{\text{inv}(\sigma)} t^{\text{maj}(\sigma)} X^\sigma.
\]
The $\Delta$ operator and the $\nabla$ operator

Given any partition $\mu \vdash n$, we can draw the Ferrers diagram (in French notation) of $\mu$ as shown in Figure 1.

![Diagram of Ferrers diagram](image)

Figure 1: The Young tableau of the partition $(7, 7, 5, 3, 3)$
The $\Delta$ operator and the $\nabla$ operator

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![Ferrers diagram](image)

Figure 1: The Young tableau of the partition $(7, 7, 5, 3, 3)$

- We let $B_\mu = \sum_{c \in \mu} q^{a'_\mu(c)} t^{l''_\mu(c)}$ and $T_\mu = \prod_{c \in \mu} q^{a'_\mu(c)} t^{l''_\mu(c)}$. 

$a_\mu(c) = \text{arm of } c$, 
$a'_\mu(c) = \text{coarm of } c$, 
$l_\mu(c) = \text{leg of } c$, 
$l'_\mu(c) = \text{coleg of } c$. 

For example, $B_{3,1} = 1 + q + q^2 + t$, $T_{3,1} = q^3 t$. 

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Given a symmetric function $f(X)$, the operator nabla $\nabla$ introduced by F. Bergeron and Garsia is defined by

$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu.$$
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- We define delta operator $\Delta f$ by $\Delta f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu$. 

Note that $\nabla = \Delta e$ on $\Lambda(n)$. 

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For example,

$$\Delta_{e_2} \tilde{H}_{3,1} = e_2[1 + q + q^2 + t] \tilde{H}_{3,1} = (q + q^2 + t + q^3 + qt + q^2 t) \tilde{H}_{3,1}$$

Note that $\nabla = \Delta_{e_n}$ on $\Lambda^{(n)}$. 

\[ \begin{array}{|c|c|c|} 
\hline 
 t & 1 & q & q^2 \\
\hline 
\end{array} \]
The $\Delta'$ operator

- We also define $\Delta'_f$ by

$$\Delta'_f \tilde{H}_\mu = f[B_\mu - 1] \tilde{H}_\mu.$$
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$$\Delta'_{e_2} \tilde{H}_{3,1} = e_2[q + q^2 + t] \tilde{H}_{3,1}$$

$$= (q^3 + qt + q^2 t) \tilde{H}_{3,1}$$
The $\Delta'$ operator

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- For example, 

$$\begin{array}{c|cc}
t & q & q^2 \\
1 & & \\
\end{array}$$ 

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- Note that $\Delta_{e_k} = \Delta'_{e_k} + \Delta'_{e_{k-1}}$ since $e_k[X + 1] = e_k[X] + e_{k-1}[X]$.

- In $\Lambda^{(n)}$, since $\Delta'_{e_n} = 0$, we have $\nabla = \Delta_{e_n} = \Delta'_{e_{n-1}}$. 

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Conjectures for $\Delta'_{e_k} \Delta_{h_{r \ n}} e_n$ 

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Definition (Dyck path)

An $n \times n$ Dyck path is a lattice path from $(0, 0)$ to $(n, n)$ consisting of east and north steps which stays above the diagonal $y = x$.

We can get an $n \times n$ parking function by labeling the cells east of and adjacent to a north step of a Dyck path.
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Figure 2: The construction of a parking function
Area of a Parking Function

**Definition (area)**

The number of full cells between the Dyck path of a parking function $PF$ and the main diagonal is denoted by $\text{area}(PF)$.

\[
\text{area}(PF) = \sum_{i=1}^{n} a_i(PF).
\]

<table>
<thead>
<tr>
<th>$i$</th>
<th>$a_i(PF)$</th>
<th>$d_i(PF)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
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<td>3</td>
<td>2</td>
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</table>

\[\text{area}(PF) = 13\]

*Figure 3: A (7, 7)-Parking Function*
Dinv of a Parking Function

Definition (dinv)

We let \( d_i(PF) = \left| \{(i, j) \mid i < j, \ a_i = a_j \text{ and } \ell_i < \ell_j \} \right| \cup \{(i, j) \mid i < j, \ a_i = a_j + 1 \text{ and } \ell_i > \ell_j \} \right|. \]

Then,

\[
dinv(PF) = \sum_{i=1}^{N} d_i(PF).
\]

Figure 3: A (7, 7)-Parking Function
Statistics of an \((n, n)\)-PF

- **word** \(\sigma\): reading cars from highest \(\rightarrow\) lowest diagonal.
  \(\sigma(PF) = 6741532.\)

- \(i\text{Des}(PF) = i\text{Des}(\sigma(PF)) = \{i \in \sigma : i + 1 \leftarrow i\}\).
  \(i\text{Des}(PF) = \{2, 3, 5\}\).

\[
\text{weight} = t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{i\text{Des}(PF)} = t^{13} q^{2} F_{2,3,5}
\]

**Figure 3:** A \((7, 7)\)-Parking Function
Let $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_n$ be two sets of $n$ variables. The ring of **Diagonal harmonics** consists of those polynomials in $\mathbb{Q}[X, Y]$ which satisfy the following system of differential equations

$$
\partial_{x_1}^a \partial_{y_1}^b f(x, y) + \partial_{x_2}^a \partial_{y_2}^b f(x, y) + \ldots + \partial_{x_n}^a \partial_{y_n}^b f(x, y) = 0,
$$

for each pair of integers $a$ and $b$, such that $a + b > 0$. 

Haiman proved that the ring of diagonal harmonics has dimension $\binom{n}{2} + 1$. 

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Conjectures for $\Delta'_{\text{ek}} \Delta_{hr} e_n$  
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$$

for each pair of integers $a$ and $b$, such that $a + b > 0$.

Haiman proved that the ring of diagonal harmonics has dimension $(n+1)^{n-1}$. 
The **bigraded Frobenius characteristic** of the $S_n$-module (under the diagonal action) of the ring of diagonal harmonics is given by $\nabla e_n$.

The classical shuffle conjecture (now the **Shuffle Theorem**) of Haglund, Haiman, Loehr, Remmel, and Ulyanov (2005) gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics:

**Theorem (Carlson-Mellit)**

For all $n \geq 0$,

$$\nabla e_n = \sum_{PF \in PF} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{Des}}(PF).$$

The theorem was proved by Carlson and Mellit in 2015.
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For all $n \geq 0$,

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The Shuffle Theorem has many generalizations. For example, the Compositional Shuffle Conjecture and the Rational Shuffle Conjecture, which are proved.
The Shuffle Theorem – generalizations

The Shuffle Theorem has many generalizations. For example, the Compositional Shuffle Conjecture and the Rational Shuffle Conjecture, which are proved.

The Delta Conjecture of Haglund, Remmel and Wilson is another well studied extension of the shuffle Theorem. The Delta Conjecture has two versions, rise version and valley version.
Definition \((\text{area})\)

The number of full cells between the Dyck path of a parking function \(PF\) and the main diagonal is denoted by \(\text{area}(PF)\).

\[
\text{area}(PF) = \sum_{i=1}^{n} a_i(PF).
\]

\(\begin{array}{|c|c|c|}
\hline
i & a_i(PF) & d_i(PF) \\
\hline
7 & 3 & 0 \\
6 & 2 & 0 \\
5 & 3 & 1 \\
4 & 2 & 1 \\
3 & 2 & 0 \\
2 & 1 & 0 \\
1 & 0 & 0 \\
\hline
\end{array}\)

\(\text{area}(PF) = 13\)

Figure 3: A \((7, 7)\)-Parking Function
The Delta Conjecture – Rise Version

Delta Conjecture, Rise Version (Haglund, Remmel and Wilson)

\[
\Delta'_{e_k} e_n = \sum_{PF \in \mathcal{P}F_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{i\text{Des}(PF)} \prod_{i \in \text{Rise}(PF)} \left(1 + \frac{z}{t^{a_i(PF)}}\right) \bigg|_{z^n - (k + 1)}.
\]

Here \(\text{Rise}(PF) = \{i | a_i(PF) > a_{i-1}(PF)\}\) is the collection of double rise of the path of the parking function \(PF\).
The Delta Conjecture – Rise Version

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\[ \Delta_{e_k} e_n = \sum_{P\in \mathcal{P}F_n} t^{\text{area}(P\text{F})} q^{\text{dinv}(P\text{F})} F_{\text{iDes}(P\text{F})} \prod_{i \in \text{Rise}(P\text{F})} \left(1 + \frac{z}{t^{a_i(P\text{F})}}\right) \bigg|_{z^{n-k-1}}. \]

Here \( \text{Rise}(P\text{F}) = \{ i | a_i(P\text{F}) > a_{i-1}(P\text{F}) \} \) is the collection of double rise of the path of the parking function \( P\text{F} \).

For example, the parking function \[
\begin{array}{c}
3 \\
2 \\
1 \\
\end{array}
\]
contributes \( t^2 q^0 F_{1,2} (1 + z/t) \big|_z \) which equals \( tF_{1,2} \) to \( \Delta'_{e_1} e_3 \).
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For example, the parking function \begin{tabular}{|c|c|} \hline 3 & 2 \\ \hline 1 & \hline \end{tabular} contributes \( t^2 q^0 F_{1,2}(1 + z/t) \bigg|_z \)

which equals \( tF_{1,2} \) to \( \Delta'_{e_1} e_3 \).

Taking the coefficient of \( z^{n-k-1} \) is like deleting \( n - k - 1 \) rows from double rise to compute area.
Dinv of a Parking Function

Definition (dinv)

We let \( d_i(PF) = \left| \{(i, j) | i < j, \ a_i = a_j \text{ and } \ell_i < \ell_j\} \right| \cup \left\{(i, j) | i < j, \ a_i = a_j + 1 \text{ and } \ell_i > \ell_j\} \right| \)

Then,

\[
dinv(PF) = \sum_{i=1}^{N} d_i(PF).
\]

**Figure 3: A (7, 7)-Parking Function**
The Delta Conjecture – Valley Version

Delta Conjecture, Valley Version (Haglund, Remmel and Wilson)

\[ \Delta'_{e_k} e_n = \sum_{\text{PF} \in \mathcal{PF}_n} \, t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} F_{i\text{Des}}(\text{PF}) \prod_{i \in \text{Val}(\text{PF})} \left(1 + \frac{z}{q^{d_i(\text{PF})+1}}\right) \bigg|_{z^{n-k-1}}. \]

Here \( \text{Val}(\text{PF}) = \{i | a_i < a_{i-1} \text{ or } a_i = a_{i-1} \text{ and } l_i > l_{i-1}\} \) is the collection of contractible valley of the path of the parking function \( \text{PF} \).
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Delta Conjecture, Valley Version (Haglund, Remmel and Wilson)

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For example, the parking function \[
\begin{array}{ccc}
3 & 2 \\
1 & & \\
\end{array}
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contributes \( t^1 q^2 F_{1,1,1} (1 + z/q) \bigg|_{z} \), which equals \( qtF_{1,1,1} \) to \( \Delta'_{e_1} e_3 \).
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contributes \[ t^1 q^2 F_{1,1,1}(1 + z/q) \bigg|_z \] which equals \( q t F_{1,1,1} \) to \( \Delta'_{e_1} e_3 \).

Taking the coefficient of \( z^{n-k-1} \) is like deleting \( n-k-1 \) rows from contractible valley and then deduct \( n-k-1 \) to compute dinv.
What is known for $\Delta'_e e_n$? $q = 0$ or $t = 0$ case

The Delta Conjecture is not yet proved, but a lot of cases are proved. We start with $q = 0$ or $t = 0$ case.
What is known for $\Delta'_{ek} e_n$? $q = 0$ or $t = 0$ case

The Delta Conjecture is not yet proved, but a lot of cases are proved. We start with $q = 0$ or $t = 0$ case.

We let $\text{Rise}_{n,k}(X; q, t)$ and $\text{Val}_{n,k}(X; q, t)$ be the RHS (the combinatorial side) of the rise version and the valley version of the conjecture. Then there are many connections of the combinatorial side with ordered multiset partition statistics distributions.
The ordered set partitions of $n$ with $k$ blocks are partitions of \{1,\ldots,n\} into $k$ ordered subsets, denoted $\mathcal{OP}_{n,k}$.

For example, $27/145/36 \in \mathcal{OP}_{7,3}$. 
The ordered set partitions of $n$ with $k$ blocks are partitions of \{1, \ldots, n\} into $k$ ordered subsets, denoted $\mathcal{OP}_{n,k}$.

For example, $27/145/36 \in \mathcal{OP}_{7,3}$.

Further, given a weak composition $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, we let $\mathcal{OP}_{\alpha,k}$ denote the set of partitions of multiset \{i^{\alpha_i} : 1 \leq i \leq n\} into $k$ ordered sets.

For example, $13/14/345 \in \mathcal{OP}_{\{2,0,2,2,1\},3}$. 
Given $\pi \in \mathcal{OP}_{\alpha,k}$, the inversion of $\pi$, denoted $\text{inv}(\pi)$ is the number of pairs $a > b$ such that $a$’s block is strictly to the left of $b$’s block and $b$ is the minimum of that block.

For example, $15/23/4$ has 2 inversions, caused by $(5,2)$ and $(5,4)$.
Given $\pi = \pi_1 \cdots \pi_k \in \mathcal{OP}_{\alpha,k}$, let $\pi^h_i$ be the $h^{th}$ smallest number in $\pi_i$. The diagonal inversion of $\pi$, denoted $\text{dinv}(\pi)$, is defined by

$$\text{dinv}(\pi) = |\{(h, i, j) : 1 \leq i < j \leq k, \pi^h_i > \pi^h_j\}| + |\{(h, i, j) : 1 \leq i < j \leq k, \pi^h_i < \pi^{h+1}_j\}|$$

For example, $\overline{2} \hat{4}/\overline{1} \hat{3} 4/2$ has 3 diagonal inversions caused by $(2, 1)$, $(4, 3)$ and $(2, 3)$. 
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For example, $\bar{2} \hat{4} / \bar{1} \hat{3} 4/2$ has 3 diagonal inversions caused by $(2, 1)$, $(4, 3)$ and $(2, 3)$.

Two other statistics called \text{maj} and \text{minimaj} are also defined on ordered multiset partitions.
Let $D_{\alpha,k}^{\text{stat}}(q) = \sum_{\pi \in \mathcal{OP}_{\alpha,k}} q^{\text{stat}(\pi)}$, Haglund, Remmel and Wilson showed that

**Theorem (Haglund, Remmel and Wilson)**

\[
\begin{align*}
\text{Rise}_{n,k}(X; q, 0)|_{M_\alpha} &= D_{\alpha,k+1}^{\text{dinv}}(q), \\
\text{Rise}_{n,k}(X; 0, q)|_{M_\alpha} &= D_{\alpha,k+1}^{\text{maj}}(q), \\
\text{Val}_{n,k}(X; q, 0)|_{M_\alpha} &= D_{\alpha,k+1}^{\text{inv}}(q), \\
\text{Val}_{n,k}(X; 0, q)|_{M_\alpha} &= D_{\alpha,k+1}^{\text{minimaj}}(q).
\end{align*}
\]

Thus, we can work on ordered set partition instead of on parking functions for the combinatorial side of the Delta Conjecture when $q$ or $t = 0$. 

The following theorem due to the work of Haglund, Remmel, Rhoades and Wilson shows that the Rise and the Valley version of the conjecture are equivalent when $q$ or $t = 0$.

**Theorem (Haglund, Remmel, Rhoades and Wilson)**

$$
\sum_{\pi \in \mathcal{O}\mathcal{P}_{\alpha,k+1}} q^{\text{minimaj}(\pi)} = \sum_{\pi \in \mathcal{O}\mathcal{P}_{\alpha,k+1}} q^{\text{dinv}(\pi)} = \sum_{\pi \in \mathcal{O}\mathcal{P}_{\alpha,k+1}} q^{\text{maj}(\pi)} = \sum_{\pi \in \mathcal{O}\mathcal{P}_{\alpha,k+1}} q^{\text{inv}(\pi)}.
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Equivalence of Rise and Valley Version when $q = 0$ or $t = 0$

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\]

As a consequence,

\[
\text{Rise}_{n,k}(X; q, 0) = \text{Rise}_{n,k}(X; 0, q) = \text{Val}_{n,k}(X; q, 0) = \text{Val}_{n,k}(X; 0, q).
\]
Haglund, Rhoades and Shimozono were able to represent $\text{Rise}_{n,k}(X; q, 0)$ (up to $q$-reverse and $\omega$ action) as the graded Frobenius character of ring $R_{n,k}$ which is a generalization of the coinvariant algebra. They have a nice expansion in dual Hall-Littlewood polynomials $Q'_\lambda(X; q)$.
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Garsia, Haglund, Remmel and Yoo proved the Delta Conjecture at $q = 0$ using the expansion in dual Hall-Littlewood basis.
Haglund, Rhoades and Shimozono were able to represent \( \text{Rise}_{n,k}(X; q, 0) \) (up to \( q \)-reverse and \( \omega \) action) as the graded Frobenius character of ring \( R_{n,k} \) which is a generalization of the coinvariant algebra. They have a nice expansion in dual Hall-Littlewood polynomials \( Q'_{\lambda}(X; q) \).

Garsia, Haglund, Remmel and Yoo proved the Delta Conjecture at \( q = 0 \) using the expansion in dual Hall-Littlewood basis.

Haglund, Rhoades and Shimozono gave a new proof of this result recently.
What else is known for $\Delta'_{ek} e_n$

- **The conjecture for** $\Delta_{e_1} e_n$ **is proved by Haglund, Remmel and Wilson.**
What else is known for $\Delta'_{e_k} e_n$

- **The conjecture for** $\Delta_{e_1} e_n$ is proved by Haglund, Remmel and Wilson.

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- **The Rise Version Delta Conjecture at** $q = 1$ **is proved by Romero.**

- **Catalan case** of the conjecture is proved by Zabrocki.
Our Problem: Conjectures for $\Delta_{e_k}^' \Delta_{h_r} e_n$

Our main focus is the combinatorics of $\Delta_{e_k}^' \Delta_{h_r} e_n$.

The problem was initially proposed by Haglund, Remmel and Wilson, who gave a rise version conjecture for $\Delta_{e_k}^' \Delta_{h_r} e_n$, whose combinatorial side was decorated parking functions with blank valleys.
Our Problem: Conjectures for $\Delta_{e_k}' \Delta_{h_r} e_n$

Our main focus is the combinatorics of $\Delta_{e_k}' \Delta_{h_r} e_n$.

The problem was initially proposed by Haglund, Remmel and Wilson, who gave a rise version conjecture for $\Delta_{e_k}' \Delta_{h_r} e_n$, whose combinatorial side was decorated parking functions with blank valleys.

We extend the conjecture into both rise version and valley version, and prove some combinatorics about the combinatorial side.
Given a path $P$, valley of $P$ is defined by
\[
\text{valley}(P) = \{i | a_i \leq a_{i-1} \text{ and } i \geq 2\}.
\]
Statistics of Parking Functions with Blank Valleys

- Given a path $P$, valley of $P$ is defined by

$$\text{valley}(P) = \{i | a_i \leq a_{i-1} \text{ and } i \geq 2\}.$$ 

- We say that a parking function has $r$ blank valley if there are $r$ valleys not receiving a label.

![Diagram of a parking function with 2 blank valleys]

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<th>$i$</th>
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Figure 5: A (7, 7)-Parking Function with 2 blank valleys
The areas $a_i$ are defined as before.

The dinvs $d_i$ are calculated by labeling the blank valleys with 0s.

$iDes(PF)$ is the iDes for the nonblank labels.

![Parking Function Diagram](image)

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Figure 5: A (7, 7)-Parking Function with 2 blank valleys
The double rise rows are defined as before.

The contractible valley rows are selected by labeling the blank valleys with 0s.

Let $\mathcal{PF}_N^\text{Blank}_r$ be the set of word parking functions of size $N$ with $r$ blank valleys.

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Figure 5: A (7, 7)-Parking Function with 2 blank valleys
Conjecture for $\Delta'_{e_k} \Delta_{h_r} e_n$ – Rise Version

For any positive integers $n$, $k$, and $r$ with $k < n$,

$$\Delta'_{e_k} \Delta_{h_r} e_n = \sum_{PF \in \mathcal{P}F_{n+r}^{\text{Blank}}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{i\text{Des}(PF)} \prod_{i \in \text{Rise}(PF)} \left(1 + \frac{z}{t^{a_i(PF)}}\right) \Bigg|_{z^{n-k-1}}.$$

**Figure 5: A (7, 7)-Parking Function with 2 blank valleys**

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weight$(PF) = t^6 q^3 F_{2,3,4}$

to $\Delta'_{e_1} \Delta_{h_2} e_5$
Conjecture for $\Delta_{e_k}^\prime \Delta_{h_r} e_n$ – Valley Version

For any positive integers $n$, $k$, and $r$ with $k < n$,

$$
\Delta_{e_k}^\prime \Delta_{h_r} e_n = \sum_{PF \in \mathcal{P}_n^{Blank}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{iDes}(PF)} \prod_{i \in \text{Val}(PF)} \left(1 + \frac{z}{q^{d_i(PF)}+1}\right) \mid_{z^n-k-1}.
$$

![Parking Function Diagram]

**Figure 5:** A $(7, 7)$-Parking Function with 2 blank valleys

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$\text{weight}(PF) = 0$ to $\Delta_{e_1}^\prime \Delta_{h_2} e_5$
We have some combinatorial progress on $q = 0$ and $t = 0$ case.

We let $\text{Rise}_{n,k,r}(X; q, t)$ and $\text{Val}_{n,k,r}(X; q, t)$ be the RHS (the combinatorial side) of the rise version and the valley version of the conjecture. Then there are many connections of the combinatorial side with ordered multiset partition (with zeros) statistics distributions.
We form sets of ordered multiset partitions.
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Given a weak composition $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, we let $\mathcal{OP}_{\alpha,k,r}$ denote the set of partitions of multiset $\{i^{\alpha_i} : 1 \leq i \leq n\} \cup 0^r$ into $k$ ordered sets, such that 0 does not show up in the last set.
Ordered Set and Multiset Partitions with zeros

We form sets of ordered multiset partitions.

Given a weak composition $\alpha = \{\alpha_1, \ldots, \alpha_n\}$,
we let $\mathcal{OP}_{\alpha,k,r}$ denote the set of partitions of multiset
$\{i^{\alpha_i} : 1 \leq i \leq n\} \cup 0^r$ into $k$ ordered sets, such that 0 does not show up in
the last set.

It is not hard to check that the four statistics, inversion, diagonal
inversion, maj and minimaj are well defined on sets $\mathcal{OP}_{\alpha,k,r}$.
Let \( D_{\alpha,k,r}^{\text{stat}}(q) = \sum_{\pi \in \mathcal{OP}_{\alpha,k,r}} q^{\text{stat}(\pi)} \), we prove using similar techniques that

**Theorem (Q. – Wilson)**

\[
\begin{align*}
\text{Rise}_{n,k,r}(X; q, 0)|_{M_\alpha} &= D_{\alpha,k+1,r}^{\text{dinv}}(q), \\
\text{Rise}_{n,k,r}(X; 0, q)|_{M_\alpha} &= D_{\alpha,k+1,r}^{\text{maj}}(q), \\
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\end{align*}
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Connection between PF’s and OMP’s

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\end{align*}
\]

Thus, we can work on ordered set partition instead of on parking functions for the combinatorial side of the \( \Delta_{e_k}^{'} \Delta_{h_r} e_n \) Conjecture when \( q \) or \( t = 0 \).
Equivalence of Rise and Valley Version when $q = 0$ or $t = 0$

We can show that the Rise and the Valley version of the conjecture are equivalent when $q$ or $t = 0$ by similar but more complicated techniques that,

Theorem (Q. – Wilson)

$$
\sum_{\pi \in \mathcal{OP}_{\alpha,k+1,r}} q^{\text{minimaj}}(\pi) = \sum_{\pi \in \mathcal{OP}_{\alpha,k+1,r}} q^{\text{dinv}}(\pi) = \sum_{\pi \in \mathcal{OP}_{\alpha,k+1,r}} q^{\text{maj}}(\pi) = \sum_{\pi \in \mathcal{OP}_{\alpha,k+1,r}} q^{\text{inv}}(\pi).
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\]

As a consequence,

\[
\text{Rise}_{n, k, r}(X; q, 0) = \text{Rise}_{n, k, r}(X; 0, q) = \text{Val}_{n, k, r}(X; q, 0) = \text{Val}_{n, k, r}(X; 0, q).
\]
If we allow 0 to appear in the last set of a multiset partition, in other word, if we allow blank valley and blank row 1, we are actually getting $h_r \perp \text{Rise}_{n,k}(X; q, t)$. 
If we allow 0 to appear in the last set of a multiset partition, in other word, if we allow blank valley and blank row 1, we are actually getting $\mathcal{R} \Delta_{h_{r}} \mathcal{R} \Delta_{n \cdot k}(X; q, t)$.

Since $\mathcal{R} \Delta_{h_{r}} \mathcal{R} \Delta_{n \cdot k}(X; q, t)$ should be symmetric and Schur positive as long as the Delta Conjecture is true, we can think about the complement
- we fix a 0 in the last part of an ordered multiset partition, or we fix a blank label in the first row.
Open Problems

About our current research on the conjectures on $\Delta'_{e_k} \Delta_{h_r} e_n$ when $q = 0$ or $t = 0$, we are interested in the following problems:
Open Problems

About our current research on the conjectures on $\Delta'_{e_k} \Delta_{hr} e_n$ when $q = 0$ or $t = 0$, we are interested in the following problems:

- Find $Q'_\lambda$-expansion of $\omega \Delta'_{e_k} \Delta_{hr} e_n|_{q=0}$. 

Open Problems

About our current research on the conjectures on $\Delta'_e \Delta_h e_n$ when $q = 0$ or $t = 0$, we are interested in the following problems:

- Find $Q'_\lambda$-expansion of $\omega \Delta'_e \Delta_h e_n|_{q=0}$.
- Can we find a module such that the graded Frobenius Character is equal to $\text{Rise}_{n,k,r}(X; q, 0)$?
Open Problems

Since the Delta Conjecture is not yet proved in the general case, there are many open problems. We collect some problems that seem to be interesting.

- Show that the Valley version Delta Conjecture is symmetric in the combinatorial side.
- Show that the two versions of the Delta Conjecture are equal.
- Since $\Delta_{hr} = \Delta_{sr}$, and $\Delta_{sr}^1 + \Delta_{sr}^{1+} - 1$ and $\Delta_{hr} = \Delta_{r}^\prime \Delta_{hr} + \Delta_{r}^\prime - 1$, we want to find some conjecture for $\Delta^\prime_{sr} \lambda$ for hook shape $\lambda$.
- Are there Square Path Conjecture analogue of the Delta Conjecture?
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- Since $\Delta_{hr}e_k = \Delta_{s,1^k} + \Delta_{s+1,1^{k-1}}$ and $\Delta_{hr}e_k = \Delta'_{ek} \Delta_{hr} + \Delta'_{e_{k-1}} \Delta_{hr}$, we want to find some conjecture for $\Delta_{s\lambda}$ for hook shape $\lambda$. 

Since the Delta Conjecture is not yet proved in the general case, there are many open problems. We collect some problems that seem to be interesting.

- Show that the Valley version Delta Conjecture is symmetric in the combinatorial side.
- Show that the two versions of the Delta Conjecture are equal.
- Since $\Delta_{hre_k} = \Delta_{s_{r,1k}} + \Delta_{s_{r+1,1k-1}}$ and $\Delta_{hre_k} = \Delta'_{e_k} \Delta_h + \Delta'_{e_{k-1}} \Delta_h$, we want to find some conjecture for $\Delta_{s_\lambda}$ for hook shape $\lambda$.
- Are there Square Path Conjecture analogue of the Delta Conjecture?
Thank You!