Conjectures for the delta operator expression
\[ \Delta'_{e_k} \Delta_{h_r} e_n \]

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January 19, 2018
The Ring of Diagonal Harmonics

Let $\mathbf{X} = x_1, x_2, \ldots, x_n$ and $\mathbf{Y} = y_1, y_2, \ldots, y_n$ be two sets of $n$ variables. The ring of Diagonal harmonics consists of those polynomials in $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ which satisfy the following system of differential equations

$$
\partial_{x_1}^a \partial_{y_1}^b \ f(x, y) + \partial_{x_2}^a \partial_{y_2}^b \ f(x, y) + \ldots + \partial_{x_n}^a \partial_{y_n}^b \ f(x, y) = 0,
$$

for each pair of integers $a$ and $b$, such that $a + b > 0$.

Haiman proved that the ring of diagonal harmonics has dimension $(n + 1)^{n-1}$. 
Partition and Tableau

- $\lambda = \lambda_1, \ldots, \lambda_k$ is a partition of $n$ if $\lambda_1 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$, written $\lambda \vdash n$.

- Ex. $\lambda \vdash 3 : (3), (2, 1), (1, 1, 1)$.

- Each partition corresponds to a Ferrers diagram. For example, $\lambda = (4, 2, 1) \vdash 7$ corresponds to

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We can fill the cells of the Ferrers diagram with integers.

- Column strict tableau:

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\begin{array}{cccc}
5 & 2 & 3 \\
2 & 1 & 1 & 3 & 4 \\
\end{array}
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- Injective tableau: $\lambda \rightarrow \mathbb{Z}_+$,

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\begin{array}{cccc}
4 & 1 & 5 \\
1 & 5 \\
2 & 6 & 2 & 4 \\
\end{array}
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Symmetric Functions

- \( S_n = \{\sigma : \sigma \text{ is a permutation of } [n]\} \) is the \( n \)th symmetric group.

- \( f(X) \in \mathbb{R}[[x]] \) is a symmetric function if \( f(X) = f(\sigma(X)) \) for any permutation \( \sigma \).

- Ex. \( f(x_1, x_2, x_3) = 3x_1x_2 + 3x_1x_3 + 3x_2x_3 + \cdots + 5x_1^2x_2 + 5x_1x_2^2 + 5x_1^2x_3 + \cdots \)

- The ring of symmetric functions has several bases: \( \{s_\lambda\}, \{e_\lambda\}, \ldots \)

- \( e_n = \sum_{i_1 < \cdots < i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \), and \( e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k} \).

- \( s_\lambda = \sum_{T \text{ a column strict tableau of shape } \lambda} X^T \).
Quasi-symmetric Functions

- $f(X) \in \mathbb{R}[[x]]$ is a **quasi-symmetric function** if for each composition $\alpha(\alpha_1, \ldots, \alpha_k)$, the coefficient of the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$.

- $F_S = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n, i_j < i_{j+1}} x_{i_1} x_{i_2} \cdots x_{i_n}$ if $j \in S$

is the **fundamental quasi-symmetric function** associated with a set $S \subset [n-1]$. 
Given any partition $\mu \vdash n$, we can draw the Ferrers diagram (in French notation) of $\mu$ as shown in Figure 1.

![Diagram of Ferrers diagram]

Figure 1: The Young tableau of the partition $(7, 7, 5, 3, 3)$

Then for each cell $c \in \mu$, we have the arm $a_\mu(c)$, the coarm $a'_\mu(c)$, the leg $l_\mu(c)$, and the coleg $l'_\mu(c)$ of $c$. 
The Macdonald polynomial $\tilde{H}_\mu(X; q, t)$ is a $q, t$-weighted symmetric function given by

$$\tilde{H}_\mu(X; q, t) = \sum_{\sigma: \mu \rightarrow \mathbb{Z}_+ \text{ injective tableau}} q^{\text{inv}(\sigma)} t^{\text{maj}(\sigma)} X^\sigma.$$ 

The symmetric function operator nabla $\nabla$ is the eigenoperator on Macdonald polynomials defined by Bergeron and Garsia where

$$\nabla \tilde{H}_\mu(X; q, t) = T_\mu \tilde{H}_\mu(X; q, t).$$

Here $T_\mu = \prod_{c \in \mu} q^{a'_{\mu}(c)} t^{l'_{\mu}(c)}$. 
Dyck Paths and Parking Functions

Definition (Dyck path)

An \( n \times n \) Dyck path is a lattice path from \((0, 0)\) to \((n, n)\) consisting of east and north steps which stays above the diagonal \( y = x \).

We can get an \( n \times n \) parking function by labeling the cells east of and adjacent to a north step of a Dyck path.

Figure 2: The construction of a parking function
Area of a Dyck Path

Definition (area)

The number of full cells between an \((n, n)\)-Dyck path \(\Pi\) and the main diagonal is denoted \(\text{area}(\Pi)\).

The collection of cells above a Dyck path \(\Pi\) forms an the Ferrers diagram (English) of a partition \(\lambda(\Pi)\).

Ex. \(\lambda(\Pi) = (3, 3, 1, 1)\),

![Diagram of a (7, 7)-Dyck path with labels for area, dinv, and diagonal.]

Figure 3: A (7, 7)-Dyck path
Definition (\(dinv\))

The \(dinv\) of an \((n, n)\)-Dyck path \(\Pi\) is given by

\[
dinv(\Pi) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c) + 1} \leq 1 < \frac{\text{arm}(c) + 1}{\text{leg}(c)} \right).
\]

**Figure 3: A \((7, 7)\)-Dyck path**
Statistics of an \((n, n)\)-PF

- \(\text{area}(PF) = \text{area}(\Pi(PF)) = 8\),
- \(\text{rank}\) of a cell is \(\text{rank}(x, y) = (n + 1)y - nx\),
- \(\text{dinv}(PF) = \sum_{\text{cars}} \chi(\text{rank}(i) < \text{rank}(j) \leq \text{rank}(i) + n) = 0\),
- word \(\sigma\): reading cars from highest \(\rightarrow\) lowest rank. \(\sigma(PF) = 52431\).
- \(\text{ides}(\sigma) = \{i \in \sigma : i + 1 \leftarrow i\}\), \(\text{pides}(\sigma)\) is the composition corresponding to \(\text{ides}(\sigma)\). \(\text{ides}(PF) = \{1, 3, 4\}\) and \(\text{pides}(PF) = \{1, 2, 1, 1\}\).

\[
\text{weight} = t^{\text{area}(PF)} q^\text{dinv}(PF) F_{\text{ides}(PF)} = t^8 q^0 F_{1,3,4}
\]

Figure 4: A \((5, 5)\)-Parking Function
Classical Shuffle Conjecture

The bigraded Frobenius characteristic of the $S_n$-module (under the diagonal action) of the ring of diagonal harmonics is given by $\nabla e_n$.

The classical shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov (2005) gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics:

Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov)

For all $n \geq 0$,

$$\nabla e_n = \sum_{PF \in \mathcal{PF}_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}.$$
Symmetric Function Side Extension — ????
Thank You!