Patterns in Ordered Set Partitions and Parking Functions

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November 16, 2017
Set Partitions

Definition (Set Partition)

A set partition $\pi$ of $[n] = \{1, \ldots, n\}$ is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \ldots, B_k$ of $[n]$ called blocks such that $\bigcup_{i=1}^{k} B_i = [n]$. We write

$$\pi = B_1/\ldots/B_k,$$

where $\min(B_1) < \cdots < \min(B_k)$. 
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$$\pi = B_1/\ldots/B_k,$$

where $\min(B_1) < \cdots < \min(B_k)$.

Example

$\pi = 134/268/57 \vdash [8]$ with parts $B_1 = \{1, 3, 4\}$, $B_2 = \{2, 6, 8\}$, and $B_3 = \{5, 6\}$. 
Ordered Set Partitions

Definition (Ordered Set Partition)

An ordered set partition with underlying set partition $\pi = B_1/\ldots/B_k$ is a permutation of the blocks of $\pi$, $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$ for some permutation $\sigma$ of $[k]$. 
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Example

\( \delta = 57/134/268 \) is an ordered set partition of \([8]\) with underlying set partition \( \pi = 134/268/57 \).
Ordered Set Partitions

- $OP_n$: the set of order set partitions of $[n]$. 

Ex. $OP_{[1,1,2,2]} = OP_{5,3:1,2,2} + OP_{5,3:2,1,2} + OP_{5,3:2,2,1}$. 

$\lambda = (1^{\alpha_1} 2^{\alpha_2} ... n^{\alpha_n})$ be a partition and $\ell(\lambda) = \sum_{i=1}^{n} \alpha_i$ denote the length of $\lambda$, then $\Omega_\lambda$ denote the set of ordered set partitions $\delta = B_1/.../B_\ell(\lambda)$ of $|\lambda|$ such that the partition induced by the sizes of the parts of $\delta = \lambda$. 

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- $OP_{n,k}$: the set of order set partitions of $[n]$ with $k$ parts.
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- \( OP_{n,k:b_1,...,b_k} = \{ B_1/\ldots/B_k \in OP_{n,k} \mid |B_i| = b_i \text{ for } \forall i \} \).
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Let $\lambda = (1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n})$ be a partition and $\ell(\lambda) = \sum_{i=1}^{n} \alpha_i$ denote the length of $\lambda$, then

- $OP[\lambda]$ denote the set of ordered set partitions $\delta = B_1/\ldots/B_{\ell(\lambda)}$ of $|\lambda|$ such that the partition induced by the sizes of the parts of $\delta = \lambda$. 

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Ordered Set Partitions

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Let \( \lambda = (1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n}) \) be a partition and \( \ell(\lambda) = \sum_{i=1}^{n} \alpha_i \) denote the length of \( \lambda \), then

- \( \text{OP}_{[\lambda]} \) denote the set of ordered set partitions \( \delta = B_1/\ldots/B_{\ell(\lambda)} \) of \(|\lambda|\) such that the partition induced by the sizes of the parts of \( \delta = \lambda \).

Ex. \( \text{OP}_{[1^12^2]} = \text{OP}_{5,3:1,2,2} + \text{OP}_{5,3:2,1,2} + \text{OP}_{5,3:2,2,1} \).
Reduction of a Sequence

**Definition (red(w))**

Given a sequence of distinct positive integers \( w = w_1 \ldots w_n \), we let let the **reduction** (or **standardization**) of the sequence \( \text{red}(w) \) denote the permutation of \([n]\) obtained from \( w \) by replacing the \( i \)-th smallest letter in \( w \) by \( i \).
Reduction of a Sequence

Definition ($\text{red}(w)$)

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Example

If $w = 4592$, then $\text{red}(w) = 2341$. 
Godbole, Goyt, Herdan, and Pudwell [GGHP, 2014] used the definition that a permutation $\sigma = \sigma_1 \ldots \sigma_j \in S_j$ occurs in an ordered set partition $\delta = B_1 / \ldots / B_k$ if and only if there exists $1 \leq i_1 < \cdots < i_j \leq k$ and $b_{i_l} \in B_{i_l}$ for $l = 1, \ldots, j$ such that $\text{red}(b_{i_1} \ldots b_{i_j}) = \sigma$.

$\delta$ avoids $\sigma$ if $\sigma$ does not occur in $\delta$. 

Example \(\delta = 57/134/268\), $213$ occurs in $\delta$ since $\text{red}(518) = 213$. But $\delta$ avoids $123$ because every element in the first part \(\{5, 7\}\) of $\delta$ is bigger than every element in the second part \(\{1, 3, 4\}\) of $\delta$.
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**Example**

\( \delta = 57/134/268 \), 213 occurs in \( \delta \) since \( \text{red}(518) = 213 \).

But \( \delta \) avoids 123 because every element in the first part \( \{5, 7\} \) of \( \delta \) is bigger than every element in the second part \( \{1, 3, 4\} \) of \( \delta \).
Pattern Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \ldots \sigma_j$ is a permutation in the symmetric group $S_j$, then
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- \(\text{OP}_n(\sigma) = \{\delta \in \text{OP}_n \mid \delta \text{ avoids } \sigma\}\),
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We let

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$\bullet \ op_{[\lambda]}(\sigma) = |OP_{[\lambda]}(\sigma)|$. 
The Word of an Ordered Set Partition

**Definition ($w(\delta)$)**

The word of $\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}$ is obtained from $\delta$ by removing all the slashes, write $w(\delta)$. 

Example

If $\delta = 57/134/268$, then $w(\delta) = 57134268$. 
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The word of \(\delta = B_{\sigma_1}/\ldots/B_{\sigma_k}\) is obtained from \(\delta\) by removing all the slashes, write \(w(\delta)\).

Example

If \(\delta = 57/134/268\), then \(w(\delta) = 57134268\).
Focus 1: Word Avoidance in Ordered Set Partitions
New Definition of Pattern Avoidance

We study an alternative notion of avoidance of permutations in ordered set partitions.
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New Definition of Pattern Avoidance

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Given an ordered set partition \( \delta = B_1/ \ldots / B_k \) of \([n]\), we say that a permutation \( \sigma = \sigma_1 \ldots \sigma_j \) in the symmetric group \( S_j \) word occurs in \( \delta \) if there exists \( 1 \leq i_1 < \cdots < i_j \leq n \) such that \( \text{red}(w_{i_1} \ldots w_{i_j}) = \sigma \) where \( w(\delta) = w_1 \ldots w_n \).
Focus 1: Word Avoidance in Ordered Set Partitions
New Definition of Pattern Avoidance

We study an alternative notion of avoidance of permutations in ordered set partitions.

Given an ordered set partition $\delta = B_1/\ldots/B_k$ of $[n]$, we say that a permutation $\sigma = \sigma_1\ldots\sigma_j$ in the symmetric group $S_j$ word occurs in $\delta$ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(w_{i_1}\ldots w_{i_j}) = \sigma$ where $w(\delta) = w_1\ldots w_n$.

Thus $\sigma$ word occurs in $\delta$ if $\sigma$ classically occurs in $w(\delta)$.

We say that an ordered set partition $\delta$ word avoids $\sigma$ if $\sigma$ does not word occur in $\delta$. 
Two Kinds of Pattern Occurrences

Example

- Ordered set partition,
Two Kinds of Pattern Occurrences

Example

▶ Ordered set partition, \[
\begin{array}{ccc}
3 & 5 & 2 \\
1 & 4 &
\end{array}
\]

of \([5] = \{1, 2, 3, 4, 5\}\).
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- Pattern 132, pattern 123
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- \[
\begin{bmatrix}
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\end{bmatrix}
\] and \[
\begin{bmatrix}
1 & 3 & 4 & 5 & 2
\end{bmatrix}
\]
Two Kinds of Pattern Occurrences

Example

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- \begin{bmatrix} 3 & 5 \\ 1 & 4 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 4 & 5 & 2 \end{bmatrix}

occurrence word occurrence
Two Kinds of Pattern Occurrences

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Word Avoidance in Ordered Set Partitions

If $\sigma = \sigma_1 \ldots \sigma_j$ is a permutation in the symmetric group $S_j$, then

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Similarly, we let

- $wop_n(\sigma) = |WOP_n(\sigma)|$,
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- $wop_{[\lambda]}(\sigma) = |WOP_{[\lambda]}(\sigma)|$. 
Word Avoidance v.s. Avoidance

Word avoidance in ordered set partitions is something in-between classical avoidance in permutations and pattern avoidance in ordered set partitions in the sense of Godbole, Goyt, Herdan, and Pudwell [GGHP, 2014].
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Example
There are four ordered set partitions of 3 in which 123 word occurs:

\[123, \frac{1}{23}, \frac{12}{3} \text{ and } \frac{1}{2}/3,\]

but 123 occurs in only one permutation of 3, namely 123, and it occurs in only one ordered partition, of [3], namely 1/2/3.
Word Avoidance v.s. Avoidance—Similarity

If $\sigma$ is the decreasing permutation $\sigma = j(j-1)\ldots21$, then $OP_{n,k;b_1,\ldots,b_k}(\sigma) = WOP_{n,k;b_1,\ldots,b_k}(\sigma)$ for all $n$, $k$, and $b_1,\ldots,b_k$. 
Word Avoidance v.s. Avoidance—Similarity

If $\sigma$ is the decreasing permutation $\sigma = j(j-1)\ldots 21$, then $OP_{n,k;b_1,\ldots, b_k}(\sigma) = WOP_{n,k;b_1,\ldots, b_k}(\sigma)$ for all $n$, $k$, and $b_1,\ldots, b_k$.

For example, in [GGHP, 2014], the authors proved that

$$op_{n,k}(12) = op_{n,k}(21) = \binom{n-1}{k-1}.$$
If $\sigma$ is the decreasing permutation $\sigma = j(j-1)\ldots 21$, then $OP_{n,k;b_1,\ldots,b_k}(\sigma) = WOP_{n,k;b_1,\ldots,b_k}(\sigma)$ for all $n$, $k$, and $b_1,\ldots,b_k$.

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Thus $wop_{n,k}(21) = op_{n,k}(21) = \binom{n-1}{k-1}$.

However, $wop_{n,k}(12) = 1$ since it is easy to see that the only ordered set partition of $[n]$ which word avoids 12 is $n/(n-1)/\ldots/2/1$. 
If \( \sigma \) is the decreasing permutation \( \sigma = j(j-1) \ldots 21 \), then \( OP_{n,k;b_1,\ldots,b_k}(\sigma) = WOP_{n,k;b_1,\ldots,b_k}(\sigma) \) for all \( n, k \), and \( b_1, \ldots, b_k \).

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However, \( wop_{n,k}(12) = 1 \) since it is easy to see that the only ordered set partition of \([n]\) which word avoids 12 is \( n/(n-1)/\ldots/2/1 \).

For another example, \( wop_{n,k}(321) = \nop_{n,k}(321) \).
Results on ordered set partitions which word avoid $\sigma$

We have several results about ordered set partitions which word avoid permutations in $S_3$. 
Results on ordered set partitions which word avoid $\sigma$

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The enumerations we are interested in are $\text{wop}_{n,k}(\sigma)$ and $\text{wop}_{[b_1^{\alpha_1}, \ldots, b_k^{\alpha_k}]}(\sigma)$, which have generating functions

$$A_\sigma(x, t) = \sum_{n \geq 0} \sum_{k \geq 0} \text{wop}_{n,k}(\sigma)x^n t^k \quad (1)$$

and

$$A_{\sigma,[b_1, \ldots, b_k]}(x, t_1, \ldots, t_k) = \sum_{\alpha_1 \geq 0, \ldots, \alpha_k \geq 0} \text{wop}_{[b_1^{\alpha_1}, \ldots, b_k^{\alpha_k}]}(\sigma) x^{\sum_{i=1}^k b_i \alpha_i} t^{\sum_{i=1}^k \alpha_i} q_1^{\alpha_1} \cdots q_k^{\alpha_k} \quad (2)$$
The Patterns 132, 213, 231, 312

Theorem (Pattern 132)

\[ A_{132}(x, t) = \frac{x + 1 - \sqrt{x^2 - 4tx - 2x + 1}}{2(1 + t)x}, \]
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\[ A_{132}(x, t) = \frac{x + 1 - \sqrt{x^2 - 4tx - 2x + 1}}{2(1 + t)x}, \]

\[ wop_{n,k}(132) = \sum_{i=0}^{n/2} \sum_{j=0}^{k-i-1} 2^{k-i-j-1} \binom{n}{j, i-j, i+1, k-i-j-1, n-k-i+j} \]
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Theorem (Pattern 132)

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$$wop_{b_1^{\alpha_1} \ldots b_s^{\alpha_s}}(132) = \frac{1}{n} \binom{k}{\alpha_1 \ldots \alpha_s} \binom{n+k}{n-1}.$$
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\[ wop_{[b_1^{\alpha_1} \ldots b_s^{\alpha_s}]}(132) = \frac{1}{n} \binom{k}{\alpha_1 \ldots \alpha_s} \binom{n + k}{n - 1}. \]
Bijection between $WOP_n(132)$ and rooted planar trees with no vertices out degree 1

It follows from the Theorem that $wop_n(132)$ is the number of rooted planar trees with $n + 1$ leaves that have no vertices of out degree 1.
Bijection between $WOP_n(132)$ and rooted planar trees with no vertices out degree 1

It follows from the Theorem that $wop_n(132)$ is the number of rooted planar trees with $n + 1$ leaves that have no vertices of out degree 1.

In fact, we can give a bijective proof of this.

\[
\begin{array}{c}
4 & 5 & 2 & 1 & 3 \\
13 & \end{array} \leftrightarrow \begin{array}{c}
v_1 & 2 & v_2 & v_3 & v_4 & v_5 & v_6 \\
13 & 45 & \end{array}
\]

Bijection between $WOP_n(132)$ and rooted planar trees with no vertices out degree 1
This also allows us to compute $A_{231}(x, t)$, $A_{213}(x, t)$, $A_{312}(x, t)$ since we have produced bijections which prove the following theorem.
Symmetry among the Patterns 132,213,231,312

This also allows us to compute $A_{231}(x, t)$, $A_{213}(x, t)$, $A_{312}(x, t)$ since we have produced bijections which prove the following theorem.

**Theorem**

For all $n$, $k$, and $b_1, \ldots, b_k$ such that $b_1 + \cdots + b_k = n$,

$$wop_{n,k}(132) = wop_{n,k}(213) = wop_{n,k}(231) = wop_{n,k}(312) \text{ and}$$

$$wop_{n,k;b_1,\ldots,b_k}(132) = wop_{n,k;b_k,\ldots,b_1}(213) = wop_{n,k;b_1,\ldots,b_k}(231) = wop_{n,k;b_k,\ldots,b_1}(312).$$
Symmetry among the Patterns $132, 213, 231, 312$

A bijection between $\mathcal{OP}_n(word, 312)$ and $\mathcal{OP}_n(word, 213)$ preserving block size composition

$$w_{op_n,k}(132) = w_{op_n,k}(213) = w_{op_n,k}(231) = w_{op_n,k}(312)$$
Symmetry among the Patterns 132,213,231,312

A bijection between $\mathcal{OP}_n(\text{word, 312})$ and $\mathcal{OP}_n(\text{word, 213})$ preserving block size composition

$$wop_n,k(132) = wop_n,k(213) = wop_n,k(231) = wop_n,k(312)$$

\begin{align*}
&\begin{array}{c}
| & | & | & | \\
3 & 4 & 5 & 1 \\
2 & | & & |
\end{array} & \Rightarrow & \begin{array}{c}
| & | & | & | \\
4 & | & | & |
\end{array} & \Rightarrow & \begin{array}{c}
| & | & | & |
\end{array} \\
&\begin{array}{c}
| & | & | & | \\
5 & 3 & | & |
\end{array} & \Rightarrow & \begin{array}{c}
| & | & | & | \\
5 & 4 & 3 & 2
\end{array}
\end{align*}

\begin{align*}
p = \{3, 2, 4, 1, 5\} \in S_n(312) & \Rightarrow p' = \{5, 3, 4, 1, 2\} \in S_n(213)
\end{align*}
Symmetry between $wop_{n,k;b_1,...,b_k}(321)$ and $wop_{n,k;b_1,...,b_k}(123)$

In [GGHP, 2014], the authors proved that

$$wop_{n,k;b_1,...,b_i,b_{i+1},...,b_k}(321) = wop_{n,k;b_1,...,b_{i+1},b_i,...,b_k}(321).$$

Thus for the permutation 321, we can essentially reduces ourselves to ordered set partitions where the size of the parts weakly increase as we read from left to right. We have proved a similar result for 123.
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\text{wop}_{n,k;b_1,...,b_i,b_{i+1},...,b_k}(321) = \text{wop}_{n,k;b_1,...,b_i+1,b_i,...,b_k}(321).
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Thus for the permutation 321, we can essentially reduce ourselves to ordered set partitions where the size of the parts weakly increase as we read from left to right. We have proved a similar result for 123.

**Theorem**

\[
\text{wop}_{n,k;b_1,...,b_i,b_{i+1},...,b_k}(123) = \text{wop}_{n,k;b_1,...,b_i+1,b_i,...,b_k}(123).
\]
Pattern 123

It is easy to see that an ordered set partition that word avoids 123 can have parts of size only 1 or 2. Then we have the following theorem.

**Theorem**

\[
A_{123,[1,2]}(x, t, q_1, q_2) = \frac{1 - \sqrt{1 - 4xt(q_1 + xq_2)}}{2tx(q_1 + xq_2)}.
\]

And for \(2k \geq n\),

\[
\text{wop}_{n,k;1^{2k-n},2^{n-k}}(123) = \frac{\text{wop}_{n,k}(123)}{\binom{k}{n-k}} = \frac{1}{k+1} \binom{2k}{k} = C_k.
\]

*Here \(C_k\) is the \(k^{\text{th}}\) Catalan number.*
Pattern 321

We also have generating function of $wop_{n,k}(321)$.

**Theorem**

$$A_{321}(x, t) = \frac{2(t + 1)(x - x^2) + t - t \sqrt{1 - 4(t + 1)(x - x^2)}}{2(t + 1)^2(x - x^2)}.$$
A parking function of size $n$ can be considered as a combination of a Dyck path on an $n \times n$ lattice and an ordered set partition.

Given any Dyck path on the $n \times n$ lattice, one creates a parking function $P$ by labeling the north steps with $1, 2, \ldots, n$ in such a way that in any column, the numbers are increasing when read from bottom to top.

\begin{center}
\begin{tikzpicture}
\draw[dashed] (0,0) grid (4,4);
\draw[blue, thick] (0,0) -- (1,1) -- (2,2) -- (2,3) -- (3,3) -- (3,4);
\end{tikzpicture}
\end{center}
Focus 2: Pattern Avoidance in Parking Functions

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Given any Dyck path on the $n \times n$ lattice, one creates a parking function $P$ by labeling the north steps with 1, 2, ..., $n$ in such a way that in any column, the numbers are increasing when read from bottom to top.

![Diagram of a parking function construction]

The construction of a parking function
Word Avoidance in Parking Functions

The underlying ordered set partition $\pi(P)$ is the ordered set partition whose parts are the labels of the vertical segments when read from left to right.

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The construction of a parking function

We say that a permutation $\sigma$ word occurs in a parking function $P$ if it word occurs in $\pi(P)$. A parking function $P$ word avoids a permutation $\sigma$ if $\sigma$ does not occur in $P$. 
Word Avoidance in Parking Functions

\( PF_n \) is the set of parking functions on the \( n \times n \) lattice. If \( \sigma = \sigma_1 \ldots \sigma_j \) is a permutation in the symmetric group \( S_j \), then

- \( PF_n(\sigma) = \{ P \in PF_n \mid \pi(P) \in WOP_n \} \),
- \( PF_{n,k}(\sigma) = \{ P \in PF_n \mid \pi(P) \in WOP_{n,k} \} \),
- \( PF_{n,k:b_1,...,b_k}(\sigma) = \{ P \in PF_n \mid \pi(P) \in WOP_{n,k;b_1,...,b_k} \} \).
Word Avoidance in Parking Functions

$PF_n$ is the set of parking functions on the $n \times n$ lattice. If $\sigma = \sigma_1 \ldots \sigma_j$ is a permutation in the symmetric group $S_j$, then

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Similarly, we let

- $pF_n(\sigma) = |PF_n(\sigma)|$,
- $pF_{n,k}(\sigma) = |PF_{n,k}(\sigma)|$,
- $pF_{n,k:b_1,\ldots,b_k}(\sigma) = |PF_{n,k;b_1,\ldots,b_k}(\sigma)|$. 
Results on Pattern Avoidance in Parking Functions

We have proved a number of results on patterns in parking functions. For example, we have the following theorem.

Theorem

\[
\text{pf}_{n,k}(123) = \frac{1}{(k+1)(n-k+1)} \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} \\
= \frac{C_k}{n-k+1} \binom{n}{k} \binom{k}{n-k}.
\]

and

\[
\text{pf}_n(123) = \sum_{k=\frac{n}{2}}^{n} \frac{1}{(k+1)(n-k+1)} \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} \\
= \sum_{k=\frac{n}{2}}^{n} \frac{C_k}{n-k+1} \binom{n}{k} \binom{k}{n-k}.
\]
Focus 1’: Adding descents to generating functions

We have also found expression for generating functions over ordered set partitions which word avoid a permutation in $S_3$ where we keep track of certain kinds of descents.
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We have also found expression for generating functions over ordered set partitions which word avoid a permutation in $S_3$ where we keep track of certain kinds of descents.

Here are three natural types of descent sets that one can define on an ordered set partition $\pi = B_1/\ldots/B_k$. Let $b_{i_{\text{min}}} = \min\{B_i\}$ and $b_{i_{\text{max}}} = \max\{B_i\}$.

(a) $\text{Des}_{\text{min}}(\pi) = \{i \mid b_{i_{\text{min}}} > b_{i+1_{\text{min}}}\}$,
(b) $\text{Des}(\pi) = \{i \mid b_{i_{\text{max}}} > b_{i+1_{\text{min}}}\}$, and
(c) $\tilde{\text{Des}}(\pi) = \{i \mid b_{i_{\text{min}}} > b_{i+1_{\text{max}}}\}$. 
Focus 1’: Adding descents to generating functions

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(a) $Des_{\min}(\pi) = \{i \mid b_{i_{\min}} > b_{i+1_{\min}}\},$

(b) $Des(\pi) = \{i \mid b_{i_{\max}} > b_{i+1_{\min}}\}$, and

(c) $\tilde{Des}(\pi) = \{i \mid b_{i_{\min}} > b_{i+1_{\max}}\}.$

We let $des_{\min}(\pi) = |Des_{\min}(\pi)|$, $des(\pi) = |Des(\pi)|$, and $\tilde{des}(\pi) = |\tilde{Des}(\pi)|$. 
Focus 1’: Adding descents to generating functions

We can find the generating function $\text{des}(\pi)$ and $\tilde{\text{des}}(\pi)$ over all ordered set partitions which word avoid $\alpha$ for any $\alpha \in S_3$, and the generating function of $\text{des}_{\min}(\pi)$ over all ordered set partitions which word avoid 132.

Example

Taking pattern $\sigma = 123$ and $\text{des}(\pi)$, we have

$$A_{\sigma, \text{des}}(x, y, t) = \sum_{n \geq 0} \sum_{k \geq 0} \sum_{\pi \in \text{WOP}_{n,k}(\sigma)} x^n t^k y^{\text{des}(\pi)}$$

$$= \frac{2t^3 x^2 (y-1)^2 y - 2t^2 x (2x+1)(y-1)y + t (2x^2 y + 2xy - 1) - 1 + (1+t) \sqrt{4t^2 x^2 y^2 - 4t^2 x^2 y - 4tx^2 y - 4txy + 1}}{2t(t+1)xy^2 (txy - tx - x - 1)}.$$
Future Work

- $wop_{n,k;b_1,\ldots,b_k}(321)$ is not known yet.
- Adding $\text{des}_{\text{min}}(\pi)$ to generating function is open for several patterns.
- Many parking function pattern avoidance problems are open.

References

A. Godbole, A. Goyt, J. Herndan, and L. Pudwell (2014)
Pattern avoidance in ordered set partitions
Thank You!
Thank You!
Definition (parking function)

Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$, and let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of $\alpha$. Then $\alpha$ is a parking function iff $b_i \leq i$. 