Rational Shuffle Conjecture

When $n = 3$ or $m = 3$

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The Ring of Diagonal Harmonics

Let $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_n$ be two sets of $n$ variables. The ring of \textbf{Diagonal harmonics} consists of those polynomials in $\mathbb{Q}[X, Y]$ which satisfy the following system of differential equations

$$\partial_{x_1}^a \partial_{y_1}^b f(x, y) + \partial_{x_2}^a \partial_{y_2}^b f(x, y) + \ldots + \partial_{x_n}^a \partial_{y_n}^b f(x, y) = 0,$$

for each pair of integers $a$ and $b$, such that $a + b > 0$.

Haiman proved that the ring of diagonal harmonics has \textbf{dimension $(n + 1)^{n-1}$}.
Partition and Tableau

- \( \lambda = \lambda_1, \ldots, \lambda_k \) is a partition of \( n \) if \( \lambda_1 \geq \ldots \geq \lambda_k \) and \( \sum_{i=1}^{k} \lambda_i = n \), written \( \lambda \vdash n \).

- Ex. \( \lambda \vdash 3 : (3), (2, 1), (1, 1, 1) \).
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- Each partition corresponds to a Ferrers diagram. For example, \( \lambda = (4, 2, 1) \vdash 7 \) corresponds to

  ![Ferrers diagram](image)

  We can fill the cells of the Ferrers diagram with integers.
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- Column strict tableau:

  $$\begin{array}{cccc}
  5 & 2 & 3 \\
  2 & 3 & 1 & 1 \\
  1 & 1 & 3 & 4 \\
  \leq
  \end{array}$$
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- Column strict tableau:

  - Injective tableau: $\lambda \rightarrow \mathbb{Z}_+$,
Symmetric Functions

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- $f(X) \in \mathbb{R}[[x]]$ is a symmetric function if $f(X) = f(\sigma(X))$ for any permutation $\sigma$.

- Ex. $f(x_1, x_2, x_3) =$
  
  $3x_1x_2 + 3x_1x_3 + 3x_2x_3 + \cdots + 5x_1^2x_2 + 5x_1x_2^2 + 5x_1^2x_3 + \cdots$
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- The ring of symmetric functions has several bases: \( \{ s_\lambda \} , \{ e_\lambda \} , \ldots \)
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- $e_n = \sum_{i_1 < \cdots < i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$, and $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}$.
Symmetric Functions

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- $s_\lambda = \sum_{\mathcal{T}} X^\mathcal{T}$. 
  \( \mathcal{T} \) a column strict tableau of shape $\lambda$.
Quasi-symmetric Functions

- $f(X) \in \mathbb{R}[[x]]$ is a quasi-symmetric function if for each composition $o(\alpha_1, \ldots, \alpha_k)$, the coefficient of the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$. 

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- \( F_S = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n, i_j < i_{j+1} \text{ if } j \in S} x_{i_1} x_{i_2} \ldots x_{i_n} \) is the fundamental quasi-symmetric function associated with a set \( S \subset [n - 1] \).
Given any partition $\mu \vdash n$, we can draw the Ferrers diagram (in French notation) of $\mu$ as shown in Figure 1.

![Diagram of a Young tableau]

**Figure 1:** The Young tableau of the partition $(7, 7, 5, 3, 3)$

Then for each cell $c \in \mu$, we have the arm $a_\mu(c)$, the coarm $a'_\mu(c)$, the leg $l_\mu(c)$, and the coleg $l'_\mu(c)$ of $c$. 
Macdonald polynomials

The Macdonald polynomial $\tilde{H}_\mu(X; q, t)$ is a $q, t$-weighted symmetric function given by

$$\tilde{H}_\mu(X; q, t) = \sum_{\sigma: \mu \rightarrow \mathbb{Z}_+ \text{ injective tableau}} q^{\text{inv}(\sigma)} t^{\text{maj}(\sigma)} X^\sigma.$$
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  \]

- The symmetric function operator nabla $\nabla$ is the eigenoperator on Macdonald polynomials defined by Bergeron and Garsia where
  \[
  \nabla \tilde{H}_\mu(X; q, t) = T_\mu \tilde{H}_\mu(X; q, t).
  \]
  Here $T_\mu = \prod_{c \in \mu} q^{a'_\mu(c)} t^{l'_\mu(c)}$. 
Dyck Paths and Parking Functions

Definition (Dyck path)

An \( n \times n \) Dyck path is a lattice path from \((0, 0)\) to \((n, n)\) consisting of east and north steps which stays above the diagonal \( y = x \).

We can get an \( n \times n \) parking function by labeling the cells east of and adjacent to a north step of a Dyck path.

![Diagram of a Dyck path and parking function](image)
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We can get an $n \times n$ parking function by labeling the cells east of and adjacent to a north step of a Dyck path.

Figure 2: The construction of a parking function
Area of a Dyck Path

Definition (area)

The number of full cells between an \((n, n)\)-Dyck path \(\Pi\) and the main diagonal is denoted \(\text{area}(\Pi)\).

The collection of cells above a Dyck path \(\Pi\) forms an the Ferrers diagram (English) of a partition \(\lambda(\Pi)\).

Ex. \(\lambda(\Pi) = (3, 3, 1, 1)\), 

Figure 3: A \((7, 7)\)-Dyck path
Dinv of a Dyck Path

Definition (dinv)

The \( \text{dinv} \) of an \((n, n)\)-Dyck path \( \Pi \) is given by

\[
\text{dinv}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c)} + 1 \leq 1 < \frac{\text{arm}(c) + 1}{\text{leg}(c)} \right).
\]

Figure 3: A \((7, 7)\)-Dyck path
Statistics of an \((n, n)\)-PF

- area\((PF)\) = area\((\Pi(PF))\) = 8,
- rank of a cell is rank\((x, y)\) = \((n + 1)y - nx\),
- dinv\((PF)\) = \(\sum_{cars} i < j \chi(rank(i) < rank(j) \leq rank(i) + n)\) = 0,
- word \(\sigma\): reading cars from highest \(\rightarrow\) lowest rank. \(\sigma(PF) = 52431\).
- ides\((\sigma)\) = \(\{i \in \sigma : i + 1 \leftarrow i\}\), pides\((\sigma)\) is the composition corresponding to ides\((\sigma)\). ides\((PF)\) = \(\{1, 3, 4\}\) and pides\((PF)\) = \(\{1, 2, 1, 1\}\).
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Figure 4: A \((5, 5)\)-Parking Function
Statistics of an \((n, n)\)-PF

- \(\text{area}(\text{PF}) = \text{area}(\Pi(\text{PF}')) = 8\),
- \(\text{rank}\) of a cell is \(\text{rank}(x, y) = (n + 1)y - nx\),
- \(\text{dinv}(\text{PF}') = \sum_{\text{cars } i < j} \chi(\text{rank}(i) < \text{rank}(j) \leq \text{rank}(i) + n) = 0\),
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\[
\text{weight} = t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} F_{\text{ides}(\text{PF})} = t^8 q^0 F_{1,3,4}
\]

Figure 4: A \((5, 5)\)-Parking Function
Classical Shuffle Conjecture

The bigraded Frobenius characteristic of the $S_n$-module (under the diagonal action) of the ring of diagonal harmonics is given by $\nabla e_n$.

The classical shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov (2005) gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics:

Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov)

For all $n \geq 0$,

$$\nabla e_n = \sum_{PF \in \mathcal{PF}_n} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}.$$
Gorsky and Negut introduced operator $Q_{m,n}$ and extended the shuffle conjecture from $\nabla e_n$ to $Q_{m,n}(-1)^n$.

The main actors on the symmetric function side of the Gorsky-Negut conjecture are the operators $D_k$ for each integer $k$, which were introduced in Garsia et al.(1999). The action of $D_k$ on a symmetric function $F[X]$ is defined as

$$D_k F[X] = F \left[ X + \frac{M}{z} \right] \sum_{i \geq 0} (-z)^i e_i[X] \bigg|_{z^k},$$

where $M = (1 - t)(1 - q)$. 
Symmetric Function Side Extension — $Q_{m,n}$ Operators

We will construct a family of symmetric function operators $Q_{a,b}$ for any pair of positive integers $(a, b)$. It will be convenient to use the notation $Q_{km, kn}$ with $(m, n)$ coprime.

- For any $n \geq 0$, set $Q_{1,n} = D_n$. 
Symmetric Function Side Extension — $Q_{m,n}$ Operators

We will construct a family of symmetric function operators $Q_{a,b}$ for any pair of positive integers $(a, b)$. It will be convenient to use the notation $Q_{km, kn}$ with $(m, n)$ coprime.

- For any $n \geq 0$, set $Q_{1,n} = D_n$.

- Then we will recursively define $Q_{m,n}$ as follows for $m > 1$. Consider the $m \times n$ lattice with diagonal $y = \frac{n}{m} x$. Let $(a, b)$ be the lattice point which is closest to and below the diagonal. Set $(c, d) = (m - a, n - b)$. We will write

$$\text{Split}(m, n) = (a, b) + (c, d).$$

Then let

$$Q_{m,n} = \frac{1}{M} [Q_{c, d}, Q_{a, b}] = \frac{1}{M} (Q_{c, d} Q_{a, b} - Q_{a, b} Q_{c, d}).$$
Symmetric Function Side Extension — $Q_{m,n}$ Operators

Figure 5 gives an example of $\text{Split}(3, 5)$.

Figure 5: The geometry of $\text{Split}(3, 5)$

$\text{Split}(3, 5) = (2, 3) + (1, 2)$ so that $Q_{3,5} = \frac{1}{M} [Q_{1,2}, Q_{2,3}]$.

The same procedure gives $Q_{2,3} = \frac{1}{M} [Q_{1,2}, Q_{1,1}]$. Therefore

$$Q_{3,5} = \frac{1}{M^2} [D_2, [D_2, D_1]] = \frac{1}{M^2} (D_2 D_2 D_1 - 2 D_2 D_1 D_2 + D_1 D_2 D_2).$$
Definition (Rational Dyck path)

An \((m, n)\)-Dyck path is a lattice paths from \((0, 0)\) to \((m, n)\) which always remains weakly above the main diagonal \(y = \frac{n}{m}x\).

The cells that are passed through by the main diagonal are marked as diagonal cells.

Figure 6: A rational Dyck path
Rational Dyck Paths

Definition (area)

The number of full cells between an \((m, n)\)-Dyck path \(\Pi\) and the main diagonal is denoted \(\text{area}(\Pi)\).

The collection of cells above a Dyck path \(\Pi\) forms the Ferrers diagram (in English notation) of a partition \(\lambda(\Pi)\). Ex.

\[\lambda(\Pi) = (3, 3, 1, 1),\]

Figure 5: A rational Dyck path
Rational Dyck Paths

Definition (pdinv)

The path dinv of an \((m, n)\)-Dyck path \(\Pi\) is given by

\[
pdinv(\Pi) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c) + 1} \leq \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c)} \right).
\]

Figure 5: A rational Dyck path
Rational Parking Functions

- \( \text{area}(PF) = \text{area}(\Pi(PF)) = 4, \)
- \( \text{rank} \) of a cell is \( \text{rank}(x, y) = my - nx, \)
- \( \text{word } \sigma: \) reading cars from highest \( \rightarrow \) lowest rank. \( \sigma(PF) = 7563412. \)
- \( \text{ides}(\sigma) = \{i \in \sigma : i + 1 \leftarrow i\}, \) \( \text{pides}(\sigma) \) is the composition set of \( \text{ides}(\sigma). \) \( \text{ides}(PF) = \{2, 4, 6\} \) and \( \text{pides}(PF) = \{2, 2, 2, 1\}. \)
Rational Parking Functions

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Figure 7: A \((5, 7)\)-parking function and the ranks of its cars
Rational Parking Functions

Definition ($\text{tdinv}$)

$$\text{tdinv}(\text{PF}) = \sum_{\text{cars } i < j} \chi(\text{rank}(i) < \text{rank}(j) < \text{rank}(i) + m).$$

In Figure 10, the pairs of cars contributing to $\text{tdinv}$ are $(1, 3)$, $(1, 4)$, $(3, 5)$, $(3, 6)$, $(4, 6)$, $(5, 7)$ and $(6, 7)$.

Figure 7: A $(5, 7)$-parking function and the ranks of its cars
Rational Parking Functions

Leven and Hicks gave a simplified formula for the \( \text{dinv} \) of a PF. Set \( \frac{0}{0} = 0 \) and \( \frac{\infty}{0} = \infty \) for all \( x \neq 0 \), then

**Definition (dinvcorr)**

\[
dinvcorr(\Pi) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1} \leq \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)} \right) \\
- \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c)} \leq \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1} \right).
\]

**Definition (dinv(PF))**

Let PF be any \((m, n)\)-parking function with underlying Dyck path \( \Pi \), then

\[
dinv(PF) = \text{tdinv}(PF) + \text{dinvcorr}(\Pi).
\]
Rational Parking Functions

- If $n > m$ then
  $$\text{dinv}(PF') = \text{tdinv}(PF) - \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c)} \leq \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1} \right).$$

- If $n = m$ then $\text{dinv}(PF) = \text{tdinv}(PF)$.

- Finally, if $n < m$ then
  $$\text{dinv}(PF') = \text{tdinv}(PF) + \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1} \leq \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)} \right).$$

![Figure 8: Types of cells that contribute to dinvcorr](image)

Figure 8: Types of cells that contribute to dinvcorr
Rational Parking Functions

Definition (ret)

The \textit{ret} of a \((km, kn)\)-parking function \(PF\) is the \textit{smallest} positive \(i\) such that the supporting path of \(PF\) goes through the point \((im, in)\).

\[\text{ret} = 2\]

\textbf{Figure 9:} The \textit{ret} of a \((9, 6)\)-parking function
Extension of Shuffle Conjecture

In 2012, Hikita defined the Hikita polynomial to extend the combinatorial side of the shuffle conjecture to rational parking functions:

\[ H_{m,n}[X; q, t] = \sum_{PF \in \mathcal{PF}_{m,n}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}[X]. \]

Then the classical shuffle conjecture of HHLRU can be restated as follows.

**Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov)**

For all \( n \geq 0 \),

\[ \nabla e_n = H_{n+1,n}[X; q, t]. \]
Rational Shuffle Conjecture

In 2013, Gorsky and Negut introduced the operator $Q_{m,n}$ and give a symmetric function expression for each coprime pair $(m, n)$ which conjecturally coincides with $H_{m,n}[X; q, t]$.

Conjecture (Gorsky-Negut)

For all pairs of coprime positive integers $(m, n)$, we have

$$Q_{m,n}(-1)^n = H_{m,n}[X; q, t].$$
Rational Shuffle Conjecture

In 2015, Garsia, Leven, Wallach and Xin extended the conjecture of Gorsky and Negut to any pair of integers \((km, kn)\):

**Conjecture (Garsia, Leven, Wallach and Xin)**

For all pairs of coprime positive integers \((m, n)\) and any positive integer \(k\), we have

\[
Q_{km, kn}(-1)^{kn} = \sum_{PF \in \mathcal{PF}_{km, kn}} \left[ ret(PF) \right] \frac{1}{t} \area(PF) t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}[X],
\]
In 2015, Carlson and Mellit proved the Classical Shuffle Conjecture that
\[ \nabla e_n = H_{n+1,n}[X; q, t] = \sum_{PF \in \mathcal{PF}_{n+1,n}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}[X]. \]
Rational Shuffle Conjecture – Solved

In 2015, Carlson and Mellit proved the Classical Shuffle Conjecture that

$$\nabla e_n = H_{n+1,n}[X; q, t] = \sum_{PF \in \mathcal{P} \mathcal{F}_{n+1,n}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}[X].$$

In April 2016, Mellit proved the Rational Shuffle Conjecture that

$$Q_{km, kn}(-1)^{kn} = \sum_{PF \in \mathcal{P} \mathcal{F}_{km, kn}} [\text{ret}(PF)] t^{\frac{1}{t}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}[X].$$
The real problem for the ring of diagonal harmonics and the $Q_{m,n}$ operators.
Find the Schur function expansions.

The real problem is to find the Schur function ($\{s_\lambda\}$) expansion of $\nabla e_n$.

Similarly, we want to find the Schur function expansion of $Q_{n,m}(-1)^n$. 

Schur Basis Expansion of Rational Shuffle Conjecture

$[n]_{q,t}$ is the $q, t$-analogue of an integer that

$$[n]_{q,t} = \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \cdots + t^{n-1}.$$
Schur Basis Expansion of Rational Shuffle Conjecture

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\[ [n]_{q,t} = \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \cdots + t^{n-1}. \]

In 2014, Leven worked out the Schur basis Expansion for both sides of the rational shuffle conjecture when \(n = 2\) and \(m = 2\) that

Theorem

*For any* \(k \geq 0\),

\[ Q_{2k+1,1} = H_{2k+1,2}[X; q, t] = [k]_{q,t}s_2 + [k + 1]_{q,t}s_{1,1} \]

*and*

\[ Q_{2,2k+1} = H_{2,2k+1}[X; q, t] = \sum_{r=0}^{k} [k + 1 - r]_{q,t}s_{2r,1} s_{2k+1-2r}. \]
Schur Basis Expansion of Rational Shuffle Conjecture

Now from the extended rational shuffle conjecture of Garsia, Leven, Wallach and Xin that

\[ Q_{km, kn}(-1)^{kn} = \sum_{PF \in \mathcal{PF}_{km, kn}} [\text{ret}(PF)]_{1 \over t} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}[X], \]

we have worked out that

\[ Q_{2k, 21} = H_{2k, 2}[X; q, t] = ([k]_q t + [k-1]_q t) s_2 + ([k+1]_q t + [k]_q t) s_{1,1} \]

and

\[ Q_{2, 2k1} = H_{2, 2k}[X; q, t] = \sum_{r=0}^{k} ([k + 1 - r]_q t + [k - r]_q t) s_{2r} 1^{2k+1-2r}. \]
Problem: the Schur basis(\(\{s_{\lambda}\}\)) Expansion of both sides.
Problem: the Schur basis(\(\{s_\lambda\}\)) Expansion of both sides.

Our main result is the Schur expansion for \((m, 3)\) case and some partial results about \((3, n)\) case.
Schur Basis Expansion of Rational Shuffle Conjecture

- Problem: the Schur basis ($\{s_\lambda\}$) Expansion of both sides.
- Our main result is the Schur expansion for $(m, 3)$ case and some partial results about $(3, n)$ case.
- We begin with the observation of $Q_{m,3}(-1)$. We take $m = 3k + 1$ for an example.
## Coefficients of $s_\lambda$ in $Q_{3k+1,3}(-1)$

<table>
<thead>
<tr>
<th>$Q_{3k+1,3}(-1)$</th>
<th>$s_\lambda$</th>
<th>$s_3$</th>
<th>$s_{21}$</th>
<th>$s_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{1,3}(-1)$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>$[1]q,t$</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
<td>$+ qt[4]q,t$</td>
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<td>$+ qt([2]q,t + [3]q,t)$</td>
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<td>$+ qt([5]q,t + [6]q,t)$</td>
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<td>$+ (qt)^3[1]q,t$</td>
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<td>$+ qt([8]q,t + [9]q,t)$</td>
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<td>$+ (qt)^2([5]q,t + [6]q,t)$</td>
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<td>$+ (qt)^3([2]q,t + [3]q,t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$+ (qt)^4[1]q,t$</td>
</tr>
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</table>
Main Result
Formula for the Coefficients of Schur function expansion when \( n = 3 \).

Theorem

Let \([s_\lambda]_{m,n}\) be the coefficient of Schur basis \( s_\lambda \) in the polynomial \( Q_{m,n}(-1) \) and the polynomial \( H_{m,n}[X; q, t] \), then

\[
[\begin{array}{c}
s_3 \\ s_{21} \\ s_{1^3}
\end{array}]_{3k+1,3} \quad = \quad \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+1]_{q,t},
\]

\[
[\begin{array}{c}
s_3 \\ s_{21} \\ s_{1^3}
\end{array}]_{3k+1,3} \quad = \quad \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t}),
\]

\[
[\begin{array}{c}
s_3 \\ s_{21} \\ s_{1^3}
\end{array}]_{3k+1,3} \quad = \quad [s_3]_{3k+4,3};
\]
Formula for the Coefficients of Schur Basis When $n = 3$

(2)

\[
\begin{align*}
[s_3]_{3k+2,3} &= \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i + 2]_q,t, \\
[s_{21}]_{3k+2,3} &= \sum_{i=0}^{k} (qt)^{k-1-i} ([3i]_q,t + [3i + 1]_q,t), \\
[s_{13}]_{3k+2,3} &= [s_3]_{3k+5};
\end{align*}
\]
(3)

\[
[s_3]_{3k,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i - 1]_{q,t} + [3i]_{q,t} + [3i + 1]_{q,t}),
\]

\[
[s_{21}]_{3k,3} = (qt)^{k+1}([3]_{q,t} + 2[2]_{q,t} + [1]_{q,t}) + \sum_{i=1}^{k-1} (qt)^{k-1-i}([3i]_{q,t} + 2[3i + 1]_{q,t} + 2[3i + 2]_{q,t} + [3i + 3]_{q,t}),
\]

\[
[s_{13}]_{3k,3} = [s_3]_{3k+3}.
\]
Example: Formula for $[s_3]_{3k+1,3}$

Theorem

The coefficient of Schur basis $s_3$ in the polynomial $Q_{3k+1,3}(-1)$ and the polynomial $H_{3k+1,3}[X; q, t]$ is

$$[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i + 1]_{q,t}$$
Symmetric Function Side
The Coefficient of Schur Basis $s_3$ in the Polynomial $Q_{3k+1,3}(-1)$

We need the following lemma from Bergeron, Garsia, Leven and Xin to prove the symmetric function side of the theorem.

Lemma

*For any positive $m, n,*

$$\nabla Q_{m,n} \nabla^{-1} = Q_{m+n,n}.\$$
Symmetric Function Side
The Coefficient of Schur Basis $s_3$ in the Polynomial $Q_{3k+1,3}(-1)$

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Lemma

For any positive $m, n$,

$$\nabla Q_{m,n} \nabla^{-1} = Q_{m+n,n}.$$  

From the lemma, we can get a recursion for $Q_{m,n}$ operator that

$$Q_{m+n,n}(-1)^n = \nabla Q_{m,n} \nabla^{-1}(-1)^n = \nabla Q_{m,n}(-1)^n, \text{ and}$$

$$Q_{3(k+1)+1,3}(-1)^n = \nabla Q_{3k+1,3} \nabla^{-1}(-1)^n = \nabla Q_{3k+1,3}(-1)^n.$$
Algebraic Proof

We first apply the operator $\nabla$ to the Schur basis $s_3$, $s_{21}$ and $s_{13}$:

$$\nabla s_3 = (qt)^2 s_{21} + (qt)^2 [2]_{q,t} s_{13},$$
$$\nabla s_{21} = (qt) [2]_{q,t} s_{21} - (qt) [3]_{q,t} s_{13},$$
$$\nabla s_{13} = s_3 + ([2]_{q,t} + [3]_{q,t}) s_{21} + (qt + [4]_{q,t}) s_{13}.$$
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\nabla s_{13} = s_3 + ([2]_{q,t} + [3]_{q,t}) s_{21} + (qt + [4]_{q,t}) s_{13}.
\]

Then we can apply $\nabla$ to the polynomial $Q_{3k+1,3}(-1)$.

\[
\nabla Q_{3k+1,3}(-1) \\
= \nabla ([s_3]_{3k+1,3} s_3 + [s_{21}]_{3k+1,3} s_{21} + [s_{13}]_{3k+1,3} s_{13}) \\
= [s_3]_{3k+1,3} \nabla s_3 + [s_{21}]_{3k+1,3} \nabla s_{21} + [s_{13}]_{3k+1,3} \nabla s_{13} \\
= [s_{13}]_{3k+1,3} s_3 \\
+ \left[ (qt)^2 [s_3]_{3k+1,3} - qt [2]_{q,t} [s_{21}]_{3k+1,3} + ([2]_{q,t} + [3]_{q,t}) [s_{13}]_{3k+1,3} \right] \\
+ \left[ (qt)^2 [2]_{q,t} [s_3]_{3k+1,3} - qt [3]_{q,t} [s_{21}]_{3k+1,3} + (qt + [4]_{q,t}) [s_{13}]_{3k+1,3} \right] \\
= [s_3]_{3k+4,3} s_3 + [s_{21}]_{3k+4,3} s_{21} + [s_{13}]_{3k+4,3} s_{13},
\]

and the recursion from \([s_\lambda]_{3k+1,3}\) to \([s_\lambda]_{3k+4,3}\) is clear that

\[ [s_3]_{3k+4,3} = [s_1^3]_{3k+1,3}, \]

\[ [s_{21}]_{3k+4,3} = (qt)^2 [s_3]_{3k+1,3} - qt [2]_q,t [s_{21}]_{3k+1,3} + ([2]_q,t + [3]_q,t) [s_1^3]_{3k+1,3}, \]

\[ [s_{1^3}]_{3k+4,3} = (qt)^2 [2]_q,t [s_3]_{3k+1,3} - qt [3]_q,t [s_{21}]_{3k+1,3} + (qt + [4]_q,t) [s_1^3]_{3k+1,3}. \]
Combinatorial Side – From $F_\alpha$ to $s_\lambda$

Hikita (2012) proved that Hikita polynomials $H_{m,n}[X; q, t]$ are symmetric (in $X$) for any coprime $m, n$. 
Combinatorial Side – From \( F_\alpha \) to \( s_\lambda \)

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Theorem (Garsia and Remmel)

Suppose that $P(X)$ is a symmetric function which is homogeneous of degree $n$ and $P(X) = \sum_{\alpha \vdash n} a_\alpha F_\alpha(X)$, Then

$$P(X) = \sum_{\alpha \vdash n} a_\alpha s_{\tilde{\alpha}}(X).$$

Here $\tilde{\alpha}$ is the composition set of $\alpha$, and $s_\alpha(X) = \frac{\Delta_\alpha(X)}{\Delta(X)}$.

This allows us to transform $H_{m,n}[X; q, t]$ into Schur function expansion.
From $F_\alpha$ to $s_\lambda$ — Straightening

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of $n$. Suppose that for some $i$, $\alpha_i < \alpha_{i+1}$ (i.e. $\alpha$ is not a partition). Then $s_\alpha = -s(\alpha_1, \ldots, \alpha_{i+1}-1, \alpha_i+1, \ldots, \alpha_k)$. This action is called straightening.

Ex. $s_2, 3, 1 = -s_3 - 1, 2 + 1, 1 = -s_2, 2, 1 = 0$. 
From $F_\alpha$ to $s_\lambda$ — Straightening

- Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of $n$. Suppose that for some $i$, $\alpha_i < \alpha_{i+1}$ (i.e. $\alpha$ is not a partition). Then $s_\alpha = -s(\alpha_1, \ldots, \alpha_{i+1}-1, \alpha_{i+1}, \ldots, \alpha_k)$. This action is called straightening.

- Repeatedly applying this procedure will eventually yield a partition or a composition $\alpha'$ such that $\alpha'_j = \alpha'_{j+1} - 1$ for some $j$. In the latter case, the straightening action yields $s_{\alpha'} = -s_{\alpha'}$, hence $s_{\alpha'} = 0$.

- Ex. $s_{2,3,1} = -s_{3-1,2+1,1} = -s_{2,3,1} = 0$.

- Ex. $s_{1,3,1} = -s_{3-1,1+1,1} = -s_{2,2,1}$.
Notation for the Coeff of $s_\lambda$

- $\mathcal{PF}_{m,n,\text{word } \sigma}$ is the set of parking functions in $\mathcal{PF}_{m,n}$ with a diagonal reading word $\sigma$, and

- $\mathcal{PF}_{m,n,\text{pides } \sigma}$ is the set of parking functions in $\mathcal{PF}_{m,n}$ with a pides $\sigma$.

We define

$$h_{m,n,\text{word } \sigma}(q, t) = \sum_{PF \in \mathcal{PF}_{m,n,\text{word } \sigma}} t^{\text{area}(PF)} q^{\text{dinv}(PF)}$$

and

$$h_{m,n,\text{pides } \sigma}(q, t) = \sum_{PF \in \mathcal{PF}_{m,n,\text{pides } \sigma}} t^{\text{area}(PF)} q^{\text{dinv}(PF)}.$$

Then $h_{m,n,\text{pides } \sigma}(q, t)$ is the coefficient of $F_{\text{Set}(\sigma)}[X]$ in $H_{m,n}[X; q, t]$, i.e.

$$H_{m,n}[X; q, t] = \sum_{\sigma \mid n} h_{m,n,\text{pides } \sigma}(q, t) F_{\text{Set}(\sigma)}[X].$$
Notation for the Coeff of $s_{\lambda}$

- For the coefficients of Schur basis of symmetric function,
- We denote the set of parking functions in $\mathcal{PF}_{m,n}$ whose pides is straightened to $\pm \sigma$ as $\mathcal{PF}_{m,n,s_{\sigma}}$.
- We define

$$[s_{\sigma}]_{m,n}(q, t) = \sum_{\text{PF} \in \mathcal{PF}_{m,n,s_{\sigma}}} t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} \frac{s_{\text{pides}(\text{PF})}}{s_{\sigma}}$$

$$= \sum_{\alpha \text{ straightened to } \sigma} h_{m,n,\text{pides } \alpha}(q, t) \frac{s_{\alpha}}{s_{\sigma}}.$$

- Then naturally $[s_{\sigma}]_{m,n}(q, t)$ is the coefficient of $s_{\sigma}$ in $H_{m,n}[X; q, t]$, i.e.

$$H_{m,n}[X; q, t] = \sum_{\sigma \vdash n} [s_{\sigma}]_{m,n}(q, t)s_{\sigma}.$$
Notation for the Coeff of $s_\lambda$

- Recall that the combinatorial side is the Hikita polynomial:

$$H_{m,3}[X; q, t] = \sum_{PF \in \mathcal{PF}_{m,3}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}[X].$$

- By the action **straightening**, we can transform it to

$$H_{m,3}[X; q, t] = \sum_{PF \in \mathcal{PF}_{m,3}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} s_{\text{pides}(PF)}[X].$$
Rational Parking Functions

- area(PF) = area(Π(PF)) = 4,
- rank of a cell is rank(x, y) = my - nx,
- word σ: reading cars from highest \(\rightarrow\) lowest rank. \(\sigma(PF) = 7563412\).
- \(\text{ides}(\sigma) = \{i \in \sigma : i + 1 \leftrightarrow i\}\), \(\text{pides}(\sigma)\) is the composition set of \(\text{ides}(\sigma)\). \(\text{ides}(PF) = \{2, 4, 6\}\) and \(\text{pides}(PF) = \{2, 2, 2, 1\}\).

Figure 10: A (5, 7)-parking function and the ranks of its cars
Combinatorial Side Proof

Any parking function \( PF \in \mathcal{PF}_{m,3} \) has 3 rows, thus has only 3 cars: 1, 2, 3. So the word \( \sigma(PF) \) can be any permutation \( \sigma \in S_3 \). Table 1 shows the \( s_{\text{spides}} \) contribution of the 6 permutations in \( S_3 \).

<table>
<thead>
<tr>
<th>( \sigma \in S_3 )</th>
<th>( s_{\text{spides}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>( s_3 )</td>
</tr>
<tr>
<td>132</td>
<td>( s_{21} )</td>
</tr>
<tr>
<td>213</td>
<td>( s_{12} = 0 )</td>
</tr>
<tr>
<td>231</td>
<td>( s_{21} )</td>
</tr>
<tr>
<td>312</td>
<td>( s_{12} = 0 )</td>
</tr>
<tr>
<td>321</td>
<td>( s_{13} )</td>
</tr>
</tbody>
</table>

Table 1: Coefficients of \( s_\lambda \) in \( Q_{3k+1,3}(-1) \)

Since there are only 3 partitions of 3: \( \{3, 21, 1^3\} \), the Hikita polynomial of \((m, 3)\) case is

\[
H_{m,3}[X; q, t] = [s_3]_{m,3}s_3 + [s_{21}]_{m,3}s_{21} + [s_{13}]_{m,3}s_{13}.
\]
Combinatorial Side Proof

<table>
<thead>
<tr>
<th>$\sigma \in S_3$</th>
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<td>$s_{13}$</td>
</tr>
</tbody>
</table>

**Table 1:** Coefficients of $s_\lambda$ in $Q_{3k+1,3}(-1)$

From the table we can see that

- $[s_3]_{m,3} = h_{m,3,\text{word } 123}$,
- $[s_{21}]_{m,3} = h_{m,3,\text{word } 132} + h_{m,3,\text{word } 231}$,
- $[s_{13}]_{m,3} = h_{m,3,\text{word } 321}$,
The combinatorics of \([s_3]_{3k+1,3}\)

We take \([s_3]_{3k+1,3}\) as an example. We will construct

\[
[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i + 1]_{q,t}.
\]

Since \([s_3]_{m,3} = h_{m,3,\text{word 123}},\) we are looking at the set of parking functions in \(\mathcal{PF}_{m,3,\text{word 123}}\).

This set \(\mathcal{PF}_{m,3,\text{word 123}}\) of parking functions can be obtained by adding cars 1, 2, 3 in a rank-decreasing way to a \(m \times 3\) Dyck path, and smaller cars can’t be put on top of bigger cars, so we have one \(\mathcal{PF} \in \mathcal{PF}_{m,3,\text{word 123}}\) on each \(m \times 3\) Dyck path with no consecutive \(u, u\) steps.
The combinatorics of $[s_3]_{3k+1,3}$

Recall that $tdinv$ is defined as

$$tdinv(PF) = \sum_{\text{cars } i<j} \chi(\text{rank}(i) < \text{rank}(j) < \text{rank}(i) + m).$$

Since the word is 123, we have $\text{rank}(1) > \text{rank}(2) > \text{rank}(3)$, so there will always be no $tdinv$ for $PF \in \mathcal{PF}_{m,3,\text{word123}}$. Since $m = 3k + 1 > n = 3$ for $k \geq 1$, the dinv correction is of the third type. We have

$$dinv(PF) = dinvcorr(PF) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1} \leq \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)} \right).$$

(c) \hspace{2cm} (d)
The combinatorics of $\mathbf{s}_3^{3k+1,3}$

The partition $\lambda(\Pi)$ correspond with the Dyck path $\Pi$ of $\mathcal{PF} \in \mathcal{PF}_{m,3}$ is at most of height 2, so the leg of cells in $\lambda(\Pi)$ can be either 0 or 1. Taking Figure 10 for reference, we have

(a) Cells in $\lambda(\Pi)$ with leg = 0 and $1 < \text{arm} < k$ contribute 1 to dinv correction, marked $\bigcirc$ in Figure 10,

(b) Cells in $\lambda(\Pi)$ with leg = 1 and $k < \text{arm} < 2k - 1$ contribute 1 to dinv correction, marked $\triangle$ in Figure 10.

![Figure 10: The dinv correction of a $(3k + 1) \times 3$ Dyck path](image)
The combinatorics of $[s_3]_{3k+1,3}$

We can count dinv correction and area according to the partition $\lambda(\Pi)$ of the path $\Pi$. Each path $\Pi$ corresponds with a partition $\lambda = (\lambda_1, \lambda_2) \subseteq \lambda_0 = (2k, k)$. Let $\text{dinvcorr}(\lambda(\Pi)) = \text{dinvcorr}(\Pi)$ and $\text{area}(\lambda(\Pi)) = \text{area}(\Pi)$, then

\[
\text{area}(\Pi) = 2k - \lambda_1 - \lambda_2,
\]

and we can also write the formula for dinv correction:

\[
\text{dinvcorr}(\lambda) = \begin{cases} 
\lambda_1 - 2 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \leq k \\
2\lambda_1 - k - 3 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \geq k + 1 \\
2\lambda_2 + k - 2 & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_1 - \lambda_2 \geq k + 1 
\end{cases}
\]
The combinatorics of $[s_3]_{3k+1,3}$

Now for $[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i+1]_{q,t}$, we construct each term $(qt)^{k-1-i} [3i+1]_{q,t}$ as a sequence of parking functions.

For each $i$, we have 3 branches of partitions (or parking functions):

- $\Lambda_1 = \{(k+i+1, k), (k+i, k-1), \ldots, (k+1, k-i)\}$,
- $\Lambda_2 = \{(2k, i), (2k-1, i-1), \ldots, (2k+1-i, 1)\}$,
- $\Lambda_3 = \{(k+1, i+1), (k, i+1), \ldots, (i+2, i+1)\}$.

- The branch $\Lambda_1$ contains $\lambda$'s such that $\lambda_1 - \lambda_2 = i + 1 \leq k$ with $\lambda_2 \geq i + 1$,
- the branch $\Lambda_2$ contains all $\lambda$'s such that $\lambda_1 - \lambda_2 = 2k - i > k$, and
- the branch $\Lambda_3$ contains $\lambda$'s such that $\lambda_2 = i + 1$ and $\lambda_1 - \lambda_2 \leq k - i$. 
The combinatorics of $[s_3]_{3k+1,3}$

$|\Lambda_1| = |\Lambda_2| + 1$, and the last partition of $\Lambda_1$ is the same as the first partition in $\Lambda_3$. So as shown in Figure 11, the construction begin with alternatively taking partitions from $\Lambda_1$ and $\Lambda_2$, ending with the last partition of $\Lambda_1$. Then continue the chain by taking partitions in $\Lambda_3$ and end the chain with the last partition $(k - i + 1, k - i)$ in $\Lambda_3$.

Figure 11: The construction of $(qt)^{k-1-i}[3i + 1]_{q,t}$
The combinatorics of $[s_3]_{3k+1,3}$

Figure 12: The construction of $[s_3]_{13,3}$
The combinatorics of $[s_{13}]_{3k+1,3}$

- $[s_{13}]_{m-3,3} = [s_3]_{m,3}$. Bijection:

- $[s_{21}]_{m,3}$ is a construction problem similar to $[s_3]_{m,3}$.
- For the case $m = 3$, we have several results about $[s_{λ}]_{3,n}$. Every equation about $[s_{λ}]_{3,n}$ implies a bijection about parking functions.
Remark about pides in \((3, n)\) Case

Remark

Let \(i < j\) be two cars in the parking function. If \(i\) appears to the left of \(j\) in the diagonal word, then the cars \(i, j\) must be in different columns.

Remark

The elements in the pides of a parking function \(PF \in \mathcal{PF}_{m,n}\) is at most \(m\).

So in \((3, n)\) case, the \(\lambda\) in \([s_\lambda]_{3,n}\) can only be of form \(3^a 2^b 1^c\) with \(3a + 2b + c = n\).
Coefficients of $s_\lambda$ in $Q_{3,3k+1}(-1)^{3k+1}$

$$Q_{3,4,1} = s_{31} + [2]_q t s_{22} + ([3]_q t + [2]_q t) s_{212} + ([4]_q t + (qt)[1]_q t) s_{14}$$

$$Q_{3,7,-1} =$$

$$s_{321} + [2]_q t s_{322} + ([3]_q t + [2]_q t) s_{3212} + ([4]_q t + (qt)[1]_q t) s_{314}$$

$$+ ([4]_q t + [3]_q t + (qt)[1]_q t) s_{231}$$

$$+ ([5]_q t + [4]_q t + [3]_q t + (qt)[2]_q t) s_{213}$$

$$+ ([6]_q t + [5]_q t + (qt) ([3]_q t + [2]_q t)) s_{215}$$

$$+ ([7]_q t + [4]_q t + [1]_q t) s_{17}$$
Combinatorial Results about $[s_\lambda]_{3,n}$

- $[s_{32a_1b}]_{3,n} = [s_{2a_1b}]_{3,n-3}$. Bijection:

- $[s_{1^3}]_{n,3} = [s_{1^n}]_{3,n}$. Bijection:
Combinatorial Results about $[s_\lambda]_{3,n}$

- $[s_{21}]_{n,3} = [s_{21}^{n-2}]_{3,n}$. Bijection:

- **Straitening action**: $\text{pides}\{\cdots 1,3\cdots\} \Rightarrow \text{pides}\{\cdots 2,2\cdots\}$ for $\mathcal{PF}_{3,n}$ is clear – an involution whose fixed points are the coefficients of $[s_{2^a1^b}]_{3,n}$.

- $[s_{2a1^b}]_{3,n} = [s_{2b1^a}]_{3,3(a+b)-n}$. Bijection:
Other Projects
— Pattern Avoidance in Ordered Set Partitions and PF’s

- Ordered set partition, \(\begin{array}{c}
3 \\
1 \\
4 \\
5 \\
2 \\
\end{array}\) of \([5]\).

- pattern 132, pattern 123

- \(\begin{array}{c}
3 \\
1 \\
4 \\
5 \\
2 \\
\end{array}\) and \(\begin{array}{c}
1 \\
3 \\
4 \\
5 \\
2 \\
\end{array}\)

\[\Rightarrow \begin{array}{c}
5 \\
4 \\
3 \\
1 \\
\end{array} \quad \Rightarrow 13252 \quad \text{or} \quad \begin{array}{c}
3 \\
1 \\
4 \\
5 \\
2 \\
\end{array}\]

We have solved the generating functions of all patterns \(\rho\) of length 3, and also solved the enumeration of number of PF’s avoiding 123.
Thank You!