Student name: ____________________

Student PID: ____________________

MATH 10A (Wildstrom)

Final Exam, June 11, 2007

Circle the section which you *attend*:

A01	A02	A03	A04	A05	A06
5PM	6PM	7PM	8PM	8AM	9AM
Kim		Chris		Neal	

This test is closed-book and closed-notes, with the exception that you are allowed one $8\frac{1}{2}'' \times 11''$ page of handwritten notes. No calculator is allowed for this test. For full credit show all of your work (legibly!), unless otherwise specified.

For the purposes of this exam, "familiar functions" includes all arithmetic operations as well as trigonometric functions, inverse trigonometric functions, base 10 logarithms, and natural logarithms. Simplification of complicated polynomial and rational expressions in results is optional, but compositions of functions and associated inverse-functions should be simplified.

The problems are in no particular order, and it is suggested that you look at all of them before beginning to answer any.

1. (10 points) *A \$10,000 investment in the Black Rose mutual fund appreciates over 2 years to \$15,000, and is expected to continue this rate of growth.*

   (a) (3 points) *Taking the time of your initial \$10,000 investment as $t = 0$, construct a function $f(t)$ indicating the value of your investment over time.*

   Since the investment grows to 150% of its initial value in two years, $f(t) = 10000(1.50)^{\frac{t}{2}}$ is a canonical form; $f(t) = 10000e^{\frac{\ln 1.5}{2}t}$ is also a useful form for this same expression (and will be used below).

   (b) (4 points) *At what time will your investment be worth \$17,000?*

$$\begin{aligned} 17000 &= 10000e^{\frac{\ln 1.5}{2}t} \\ 1.7 &= e^{\frac{\ln 1.5}{2}t} \\ \ln 1.7 &= \frac{\ln 1.5}{2}t \\ \frac{2\ln 1.7}{\ln 1.5} &= t \end{aligned}$$

(c) (3 points) *After two years, what is the instantaneous rate of change in the value of your investment? Specify the units in which this quantity is measured.*

Since $f(t) = 10000e^{\frac{\ln 1.5}{2}t}$, $f'(t) = 10000 \cdot \frac{\ln 1.5}{2}e^{\frac{\ln 1.5}{2}t} = 5000(\ln 1.5)e^{\frac{\ln 1.5}{2}t}$. Thus, the rate of change in the investment's value after two years is $f(2) = 5000(\ln 1.5)e^{\ln 1.5} = 5000(\ln 1.5)(1.5) = 7500 \ln 1.5$. This quantity is in dollars per year, and represents the yearly return on investment.

2. (10 points) *Determine the following limits:*

(a) (4 points) *Using the difference quotient, find the derivative with respect to x of $2x^2 - 3x + 2$. Solutions determined without the difference quotient are not acceptable, and L'Hôpital's rule may not be used for this problem.*

$$\begin{aligned}
\frac{d}{dx}(2x^2 - 3x + 2) &= \lim_{h\to 0} \frac{\left[2(x+h)^2 - 3(x+h) + 2\right] - (2x^2 - 3x + 2)}{h} \\
&= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 2 - 2x^2 + 3x - 2}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 3h}{h} \\
&= \lim_{h\to 0} 4x + 2h - 3 \\
&= 4x - 3
\end{aligned}$$

(b) (3 points) *Find* $\lim_{s\to 3} \frac{\ln(s-2)}{s^2-2s-3}$.

The numerator and denominator of this limit both evaluate at $s = 3$ to zero, so L'Hôpital's rule can be applied to give $\lim_{s\to 3} \frac{\ln(s-2)}{s^2-2s-3} = \lim_{s\to 3} \frac{\frac{1}{s-2}}{2s-2}$. In this latter expression, the limit's numerator and denominator evaluate to nonzero real numbers, so since the expressions in the numerator and denominator are continuous, we may simply apply the expression *at* s=3 to get $\lim_{s\to 3} \frac{\frac{1}{s-2}}{2s-2} = \frac{\frac{1}{1}}{1} = 1$.

(c) (3 points) *Find* $\lim_{q\to -\infty} \frac{2q^3-q^2}{q^4-1}$.

This may be solved with L'Hôpital's rule, or with a dominance-based argument. Three successive applications of L'Hôpital's rule yield $\lim_{q\to -\infty} \frac{12}{24q}$; a dominance argument yields $\lim_{q\to -\infty} \frac{2q^3}{q^4}$, and in both cases the growth of the denominator outstrips the numerator, so that as q becomes extremely negative, this expression approaches zero.

3. (10 points) *Consider the function* $g(x) = \frac{3x^2+x-2}{x^2-x-6}$.

(a) (3 points) *Identify zeroes, vertical asymptotes, and long-term behavior on both sides of this function. Clearly label which is which.*

Factoring the numerator and denominator, $g(x) = \frac{(3x-2)(x+1)}{(x-3)(x+2)}$, so the zeroes of this function are at $x = \frac{2}{3}$ and $x = -1$, while the vertical asymptotes occur when the denominator is zero, at $x = 3$ and $x = -2$. Note: the quadratic formula could also be used too find the zeroes of the numerator and denominator.

(b) (5 points) *Identify the critical points of this function, and indicate whether each is a local maximum, local minimum, or neither.*

Using the quotient rule,

$$\begin{aligned} g'(x) &= \frac{\frac{d}{dx}(3x^2+x-2)(x^2-x-6) - (3x^2+x-2)\frac{d}{dx}(x^2-x-6)}{(x^2-x-6)^2} \\ &= \frac{(6x+1)(x^2-x-6) - (3x^2+x-2)(2x-1)}{(x^2-x-6)^2} \\ &= \frac{(6x^3-5x^2-37x-6) - (6x^3-x^2-5x+2)}{(x^2-x-6)^2} \\ &= \frac{-4x^2-32x-8}{(x^2-x-6)^2} \\ &= \frac{-4}{(x^2-x-6)^2}(x^2+8x+2) \end{aligned}$$

While the zeroes of the denominator oare technically critical points, they are not local extrema, since the function is in fact discontinuous there. Thus, the only extrema we need worry about are the zeroes of x^2+8x+2. These occur at $\frac{-8\pm\sqrt{64-8}}{2} = -4 \pm \sqrt{14}$. Since $g'(x)$ is a quotient of $(-4)(x^2-x-6)$ and a known positive number, $g'(x)$ has the opposite sign as x^2-x-6. Thus it is positive between the zeroes and negative off to the sides; $-4-\sqrt{14}$ is a transition from negative to positive and is thus a minimum; $-4+\sqrt{14}$ is the opposite and thus a maximum.

(c) (2 points) *Which if any of the critical points identified above are global maxima or global minima? Show work or explain.*

None of them are global, because the asymptotes induce arbitrarily large and arbitrarily small values of $g(x)$; thus, this function has no global maximum or minimum.

4. (10 points) *Given $f(t) = (\arcsin t)e^{\tan(t^2)}$, find $f'(t)$.*

We perform the product rule to find that a subproblem still needs to be solved:

$$\begin{aligned} f'(t) &= \left(\frac{d}{dt}\arcsin t\right)e^{\tan(t^2)} + (\arcsin t)\frac{d}{dt}e^{\tan(t^2)} \\ &= \frac{e^{\tan(t^2)}}{\sqrt{1-t^2}} + (\arcsin t)\frac{d}{dt}e^{\tan(t^2)} \end{aligned}$$

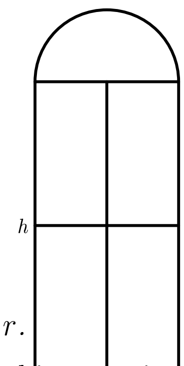
We determine $y = e^{\tan(t^2)}$ using the chain rule. Let $u = t^2$ and $v = \tan u$, so that $y = e^v$. Then

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dv}\frac{dv}{du}\frac{du}{dt} \\ &= (e^v)(\sec^2 u)(2t) \\ &= e^{\tan(t^2)} \sec^2(t^2)(2t) \end{aligned}$$

And putting these together,

$$f'(t) = \frac{e^{\tan(t^2)}}{\sqrt{1-t^2}} + (\arcsin t)(2t)\sec^2(t^2)e^{\tan t^2}$$

5. (10 points) *You are constructing a window with the shape shown here of a rectangle of height h surmounted by a semicircle of radius r. You must construct the exterior edge of the window, as well as the three internal struts shown on the figure, using only 21 feet of edging. Your goal is to maximize the area of the window.*



   (a) (3 points) *Using the knowledge that the circumference of a semicircle or radius r is πr, determine the total amount of edging used in terms of r and h, and then put h in terms of r.*

   The total amount of edging used is $6r + \pi r + 3h$. Thus, since this quantity must equal 21, algebraic manipulation yields $h = 7 - \frac{6+\pi}{3}r$.

   (b) (5 points) *Using the knowledge that the area of a semicircle is $\frac{1}{2}\pi r^2$, find an expression for the area of the window and the value of r which maximizes it.*

   The area of this entire window is $2rh + \frac{1}{2}\pi r^2$. Substituting in the known expression for h, we get that

$$A(r) = 2r(7 - \frac{6+\pi}{3}r) + \frac{1}{2}\pi r^2 = 14r - \frac{\pi + 24}{6}r^2$$

   so

$$A'(r) = 14 - \frac{\pi + 24}{3}r$$

   and solving for $A'(r) = 0$ gives that $r = \frac{14 \cdot 3}{\pi + 24}$.

   (c) (2 points) *What choices of h and r allow you to construct the largest window possible? You need not simplify your expression for h.*

   The value of r determined above is maximizing; we determine h from it by plugging back into the expression $h = 7 - \frac{6+\pi}{3}r$ to get $h = y - \frac{(6+\pi)42}{3(\pi+24)}$.

6. (10 points) *Let* $f(x) = 2\sqrt{4-x^2}$, $g(t) = (2-3t)^{1/3}$, *and* $h(\theta) = \left|\theta^3 - \theta^2\right|$. *In answering the questions below, indicate which answer is associated with each function.*

   (a) (3 points) *What is the domain of each of* f, g, *and* h*?*

   They are $[-2,2]$, $(-\infty,\infty)$, and $(-\infty,\infty)$ respectively. The only problem restricting our domain is the possibility of a negative argument for the square-root function.

   (b) (3 points) Where are each of f, g, and h continuous?

   They are $(-2,2)$, $(-\infty,\infty)$, and $(-\infty,\infty)$. Note that the endpoints of the domain of f are not in fact points of continuity, since they only have limits on one side.

   (c) (3 points) Where are each of f, g, and h differentiable?

   They are $(-2,2)$, $t \neq \frac{2}{3}$, and $\theta \neq 1$. The cube-root function goes vertical, leading to non-differentiability, and when $\theta^3 - \theta^2$ changes sign, its derivative "kinks".

7. (10 points) *Consider the relationship* $y^2 + 6y = x^3 - 7x^2 + 7x + 6$. *This is a function of a sort used extensively in cryptography, called an elliptic curve.*

   (a) (5 points) *Find* $\frac{dy}{dx}$ *in terms of* x *and* y.

   Using implicit differentiation, $2yy' + 6y' = 3x^2 - 14x + 7$, so rearranging, $y' = \frac{3x^2-14x+7}{2y+6}$.

   (b) (5 points) Find the equation of the line tangent to the curve at $(7,5)$.

   From above, $y' = \frac{3x^2-14x+7}{2y+6} = \frac{56}{16} = \frac{7}{2}$, so we want a line with slope $\frac{7}{2}$ passing through $(7,5)$. In point-slope form,

$$(y-5) = \frac{7}{2}(x-7)$$

   In slope-intercept form,

$$y = \frac{7}{2}x - \frac{39}{2}$$