Exercise 1: \( m = m_1 = 5, \alpha = 2\% \)

1) Assuming the costs of the repairs are Gaussian, the C.I for \( \frac{\sigma^2}{\nu_1} \) has the form

\[
\left[ \frac{S_1^2}{S_2^2} \cdot F_{\frac{\alpha}{2}, m-1, m-1} \right. \left. \frac{S_2^2}{S_1^2} \cdot F_{1-\frac{\alpha}{2}, m-1, m-1} \right]
\]

\[
= \left[ \frac{19.06^2}{14.80^2} \cdot \frac{11}{11}, \frac{19.06^2}{14.80^2} \cdot 11 \right]
\]

\[
= [0.1638, 19.82]
\]

2) We are 98\% confident that the true ratio of the variances lies in the interval \([0.1638, 19.82]\).

Since the confidence interval includes 1, it is possible that the two populations have the same variance.
Exercise 2

Suppose that we would like to test whether the opinions changed after the fire by using a $X^2$ test. However, the i.i.d sample consisted of pairs of opinions of 100 people $(X^1_i, X^2_i)$, ..., $(X^1_{100}, X^2_{100})$ where $(X^1_i, X^2_i)$ features the i'th person’s opinions before $(X^1_i)$ and after $(X^2_i)$ the fire.

Hence, the correct contingency table corresponding to the above data and satisfying the independence assumption of the $X^2$ test would be

<table>
<thead>
<tr>
<th></th>
<th>Sat. before</th>
<th>Sat. after</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sat. before</td>
<td>70</td>
<td>10</td>
</tr>
<tr>
<td>Sat. after</td>
<td>2</td>
<td>18</td>
</tr>
</tbody>
</table>

We now perform a $X^2$ test of independence (or homogeneity). The expected contingency table under the null is:

<table>
<thead>
<tr>
<th></th>
<th>Sat. before</th>
<th>Sat. after</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sat. before</td>
<td>57.6</td>
<td>22.4</td>
</tr>
<tr>
<td>Sat. after</td>
<td>14.4</td>
<td>5.6</td>
</tr>
</tbody>
</table>
we see that all the cells of the expected contingency table are \( \geq 5 \), so we can indeed apply a \( \chi^2 \) test.

We compute

\[
\chi^2 = \frac{(70 - 57.6)^2}{57.6} + \frac{(10 - 22.4)^2}{22.4} + \frac{(2 - 16.4)^2}{14.4} + \frac{(18 - 5.6)^2}{5.6}
\]

\[
\approx 47.6
\]

On the statistical table of the \( \chi^2 \) distribution with \( df = (11 - 1)(1 - 1) = 1 \) degrees of freedom, we read that \( \chi^2_{0.01, 1} = 6.635 < 47.6 \). Hence, the \( p \)-value of this test is \( < 0.01 = 1\% \), so we reject the null.

It appears that the opinions changed after the fire.

**Exercise 3**

1) From the Central limit theorem, \( \sqrt{n} (\hat{\theta} - \theta) \xrightarrow{d} N(0, p(1-p)) \), since \( p(1-p) = \text{Var}(X_1) \).

\( h: \theta \rightarrow \theta(1-\theta) \) is differentiable with differential \( h'(\theta) = 1 - 2\theta \).

From the Delta-method, we get

\[
\sqrt{n} (h(\hat{\theta}) - h(\theta)) \xrightarrow{d} N(0, h'(\theta) \cdot p(1-p) \cdot h'(\theta)^T)
\]

In other words,

\[
\sqrt{n} \left( \hat{\theta}(1-\hat{\theta}) - \theta(1-\theta) \right) \xrightarrow{d} N(0, (1-2\theta)^2 p(1-p))
\]
2) We can reformulate 1) as

\[
\frac{\sqrt{n} \left( \hat{p}(1-\hat{p}) - p(1-p) \right)}{1-2\hat{p}\sqrt{p(1-p)}} \xrightarrow{\mathbb{D}} N(0,1), \quad n \to \infty,
\]

where we used that \( \hat{p} \in \{0, \frac{1}{2}, 1\} \) to get \( 1-2\hat{p}\sqrt{p(1-p)} \neq 0 \).

On the other hand, the law of large numbers states that \( \hat{p} \xrightarrow{a.s.} p \).

As a consequence, \( \frac{\sqrt{n} \left( \hat{p}(1-\hat{p}) - p(1-p) \right)}{1-2\hat{p}\sqrt{p(1-p)}} \xrightarrow{a.s.} 1-2p\sqrt{p(1-p)} \).

From Slutsky's lemma, we then obtain that

\[
\frac{\sqrt{n} \left( \hat{p}(1-\hat{p}) - p(1-p) \right)}{1-2\hat{p}\sqrt{p(1-p)}} \xrightarrow{\mathbb{D}} N(0,1), \quad n \to \infty.
\]

Let \( \tilde{z}^* = \tilde{z}_{1-\alpha/2} \) denote the quantile of order \( 1-\frac{\alpha}{2} \) of \( N(0,1) \). The above convergence yields that

\[
P( -\tilde{z}^* < \frac{\sqrt{n} \left( \hat{p}(1-\hat{p}) - p(1-p) \right)}{1-2\hat{p}\sqrt{p(1-p)}} < \tilde{z}^* ) \xrightarrow{n \to \infty} 1 - \alpha,
\]

or equivalently

\[
P \left( p(1-p) \in I_{\alpha} \right) \xrightarrow{n \to \infty} 1 - \alpha,
\]
where
\[
\mathcal{I}(\theta) = \left[ \hat{\theta}(1 - \hat{\theta}) - \frac{1}{n} \left( \frac{1 - 2\hat{\theta}}{\sqrt{n}} \right) \hat{\theta}(1 - \hat{\theta}) + \frac{1}{n} \left( \frac{1 - 2\hat{\theta}}{\sqrt{n}} \right)^2 \right].
\]

Exercise 4
We write \( \sigma = (\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3) \).

The likelihood is the product of all \( m = (m_1 + m_2 + m_3) \) normal densities

\[
L(\theta, \sigma) = \frac{1}{(2\pi)^{\frac{m}{2}}} \frac{1}{\sqrt{m_1 m_2 m_3}} \exp\left( -\frac{1}{2} \sum_{i=1}^{m_1} \frac{(x_i - \mu_1)^2}{\sigma_1^2} - \frac{1}{2} \sum_{i=1}^{m_2} \frac{(x_i - \mu_2)^2}{\sigma_2^2} - \frac{1}{2} \sum_{i=1}^{m_3} \frac{(x_i - \mu_3)^2}{\sigma_3^2} \right).
\]

1) Under \( H_1 \) (unrestricted), the MLE's for the parameters are

\[
\hat{\mu}_1 = \bar{x}_{m_1}, \quad \hat{\mu}_2 = \bar{x}_{m_2}, \quad \hat{\mu}_3 = \bar{x}_{m_3}, \quad \hat{\sigma}_1^2 = \frac{1}{m_1} \sum_{i=1}^{m_1} (x_i - \bar{x}_{m_1})^2,
\]

\[
\hat{\sigma}_2^2, \quad \hat{\sigma}_3^2 \text{ defined similarly.}
\]

Under \( H_0 \), \( \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2 \) and the MLE's are

\[
\hat{\mu}_1 = \bar{x}_{m_1}, \quad \hat{\mu}_2 = \bar{x}_{m_2}, \quad \hat{\mu}_3 = \bar{x}_{m_3}, \quad \hat{\sigma}_1^2 = \frac{m_1 \hat{\sigma}_1^2 + m_2 \hat{\sigma}_2^2 + m_3 \hat{\sigma}_3^2}{m}.
\]
Plugging in these values in the likelihood ratio, we get

\[ \Lambda = \frac{\sup_{\theta | \nu_1 = \nu_2 = \nu_3} L(\theta, x)}{\sup_0 L(\theta, x)} \]

\[ = \frac{L\left( (\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3), x \right)}{L\left( (\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3), x \right)} \]

\[ = \frac{\frac{m_1 \cdot \nu_1}{\nu_2 \cdot \nu_3}}{m_2 \cdot \nu_3} \]

2) Hence, \[ \dim \mathcal{C} = \dim \left( \mathbb{R}^3 \times \mathbb{R}_{>0}^3 \right) = 6 \]

\[ \dim \mathcal{C}_0 = \dim \left( \mathbb{R}^3 \times \{(\nu_1, \nu_2, \nu_3), \nu > 0\} \right) = 4 \]

Hence, from Wilks, if \( m_1, m_2 \text{ and } m_3 \) are large, \( -2 \log \Lambda \) is approximately \( \chi^2 \) with \( 6 - 4 = 2 \) degrees of freedom. So the rejection region is

\[ -2 \log \Lambda \geq \chi^2_{0.05, 2} = 5.99 \]