Exercise 1

Suppose $X_1, \ldots, X_n$ are i.i.d. with density

$$f_{\mu, \sigma}(x) = p \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} + (1-p) \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $p \in (0, 1)$ is known, and $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ is unknown.

1. Show that the maximum likelihood estimators for $\mu$ and $\sigma$ do not exist.
2. How would you estimate $\mu$ and $\sigma$? What are the asymptotic distributions of your estimates?

Exercise 2

Consider testing the null hypothesis that $k$ Poisson means are equal,

$$H_0 : \lambda_1 = \ldots = \lambda_k$$

against $H_1 : \text{not all means are equal},$

using independent random samples of size $n_i$ from Poisson distributions with means $\lambda_i$, for $i = 1, 2, \ldots, k$.

Use the likelihood ratio method to test this with asymptotic level $\alpha \in (0, 1)$.

Exercise 3

Suppose that a random sample of size 30, $X_1, X_2, \ldots, X_{30}$ is drawn from a Uniform distribution on an interval $(1 - \theta, 2 + \theta)$. This distribution is defined through the density function

$$f_\theta(x) = \frac{1}{1 + 2\theta} 1_{(1-\theta,2+\theta)}(x).$$

1. Set up a large sample sign test (by defining a critical region or how to reject the test) for deciding whether or not the 70%-th percentile of the $X$-distribution is equal to 2, with $\alpha = 5\%$.

2. Given the data below, compute the exact $p$-value of the test in part 1.

$$1.42, -1.27, -0.33, -0.18, 1.56, 0.003, -0.17, -0.58, 0.44, -0.54, -1.65, -1.43, -0.08, 0.54, -1.35, 1.54, 2.30, -2.43, 0.47, 0.68$$

The data is represented in a histogram and qq-plot below.

![Histogram and QQ-plot](image)

Figure 1: Data representation for part 2.

3. Compute the asymptotic $p$-value for the test in part 1 using the data from part 2.
4. With what probability will your procedure commit a Type II error if 3 is the true 70%-th percentile?