Solutions for Basic Counting and Listing

CL-1.1 This is a simple application of the Rules of Sum and Product.

(a) Choose a discrete math text OR a data structures text, etc. This gives $5+2+6+3=16$.

(b) Choose a discrete math text AND a data structures text, etc. This gives $5\times 2\times 6\times 3=180$.

CL-1.2 We can form n digit numbers by choosing the leftmost digit AND choosing the next digit AND $\cdots$ AND choosing the rightmost digit. The first choice can be made in 9 ways since a leading zero is not allowed. The remaining $n-1$ choices can each be made in 10 ways. By the Rule of Product we have $9\times 10^{n-1}$. To count numbers with at most n digits, we could sum up $9\times 10^{k-1}$ for $1\le k\le n$. The sum can be evaluated since it is a geometric series. This does not include the number 0. Whether we add 1 to include it depends on our interpretation of the problem's requirement that there be no leading zeroes. There is an easier way. We can pad out a number with less than n digits by adding leading zeroes. The original number can be recovered from any such n digit number by stripping off the leading zeroes. Thus we see by the Rule of Product that there are 10^n numbers with at most n digits. If we wish to rule out 0 (which pads out to a string of n zeroes), we must subtract 1.

CL-1.3 For each element of S you must make one of two choices: "x is/isn't in the subset." To visualize the process, list the elements of the set in any order: $a_1, a_2, \ldots, a_{|S|}$. We can construct a subset by

including a_1 or not AND

including a_2 or not AND

$\cdot \quad \cdot \quad \cdot$

including $a_{|S|}$ or not.

CL-1.4 (a) By the Rule of Product, we have $9\times 10\times\cdots\times 10=9\times 10^{n-1}$.

(b) By the Rule of Product, we have 9^n.

(c) By the Rule of Sum, (answer)$+9^n=9\times 10^{n-1}$ and so the answer is $9(10^{n-1}-9^{n-1})$

CL-1.5 (a) This is like the previous exercise. There are 26^4 4-letter strings and there are $(26-5)^4$ 4-letter strings that contain on vowels. Thus we have 26^4-21^4.

(b) We can do this in two ways:
First way: Break the problem into 4 problems, depending on where the vowel is located. (This uses the Rule of Sum.) For each subproblem, choose each letter in the list and use the Rule of Product. We obtain one factor equal to 5 and three factors equal to 21. Thus we obtain 5×21^3 for each subproblem and $4\times 5\times 21^3$ for the final answer.
Second way: Choose one of the 4 positions for the vowel, choose the vowel and choose each of the 3 consonants. By the Rule of Product we have $4\times 5\times 21\times 21\times 21$.

CL-1.6 The only possible vowel and consonant pattern satisfying the two nonadjacent vowels and initial and terminal consonant conditions is CVCVC. By the Rule of Product, there are $3\times 2\times 3\times 2\times 3=108$ possibilities.

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CL-1.7 To form a composition of n, we can write n ones in a row and insert either "+" or "," in the spaces between them. This is a series of 2 choices at each of $n-1$ spaces, so we obtain 2^{n-1} compositions of n. The compositions of 4 are

$$4 = 3+1 = 2+2 = 2+1+1 = 1+3 = 1+2+1 = 1+1+2 = 1+1+1+1.$$

The compositions of 5 with 3 parts are

$$3+1+1 = 2+2+1 = 2+1+2 = 1+3+1 = 1+2+2 = 1+1+3.$$

CL-1.8 The allowable letters in alphabetic order are A, I, L, S, and T. There are 216 words that begin with L, and the same number that begin with S, and with T. The word we are asked to find is the last one that begins with L. Thus the word is of the form $LVCVCC$, $LVCCVC$, or $LCVCVC$. Since all of the consonants in our allowable-letters list come after the vowels, we want a word of the form $LCVCVC$. We need to start off $LTVCVC$. The next letter, a vowel, needs to be I (bigger than A in the alpahbet). Thus we have $LTICVC$. Continuing in this way we get $LTITIT$. The next name in dictionary order starts off with S and is of the form $SVCVCC$. We now choose the vowels and consonants as small as possible: $SALALL$. But, this word doesn't satisfy the condition that adjacent consonants must be different. Thus the next legal word is $SALALS$.

CL-1.9 The ordering on the C_i is as follows:

$$C_1 = ((2,4),(2,5),(3,5)) \quad C_2 = (\mathrm{AA},\mathrm{AI},\mathrm{IA},\mathrm{II})$$

$$C_3 = (\mathrm{LL},\mathrm{LS},\mathrm{LT},\mathrm{SL},\mathrm{SS},\mathrm{ST},\mathrm{TL},\mathrm{TS},\mathrm{TT}) \quad C_4 = (\mathrm{LS},\mathrm{LT},\mathrm{SL},\mathrm{ST},\mathrm{TL},\mathrm{TS}).$$

The first seven are

(2,4)(AA)(LL)(LS), (2,4)(AA)(LL)(LT), (2,4)(AA)(LL)(SL),

(2,4)(AA)(LL)(ST), (2,4)(AA)(LL)(TL),

(2,4)(AA)(LL)(TS), (2,4)(AA)(LS)(LS).

The last 7 are

(3,5)(II)(TS)(TS), (3,5)(II)(TT)(LS), (3,5)(II)(TT)(LT),

(3,5)(II)(TT)(SL), (3,5)(II)(TT)(ST),

(3,5)(II)(TT)(TL), (3,5)(II)(TT)(TS).

The actual names can be constructed by following the rules of construction from these strings of symbols (e.g, (3,5)(II)(TT)(LS) says place the vowels II in positions 3,5, the nonadjacent consonants are TT and the adjacent consonants are LS to get LSITIT).

CL-2.1 (a) We can arrange n people in $n!$ ways. Use $n=7$ here and $n=6$ in (d).

(b) Arrange b boys ($b!$ ways) AND arrange g girls ($g!$ ways) AND choose which list comes first (2 ways). Thus we have $2(b!\,g!)$. Here $b=3$ and $g=4$ and the answer is 288. In (d), $b=g=3$ and the answer is 72.

(c) As in (b), we arrange the girls and the boys separately, AND then we interleave the two lists as BGBGBGB. Thus we get $4!\ 3! = 144$. For (d) we can interleave as BGBGBG or as GBGBGB and so we get $2(3!\ 3!) = 72$.

(e) For (a) we have the circular list discussed in the text and the answer is therefore $n!/n = (n-1)!$.
For (b), note that each circular list gives two ordinary lists — one starting with the girls and the other with the boys. Hence the answer is $2(b!\ g!)/2 = b!\ g!$. For the two problems we have $4!\ 3! = 144$ and $3!\ 3! = 36$.
For (c), it is impossible since $b < g$ forces two girls to sit together. For the variant in (d), we have $b = g$ and so circular lists are possible. As in the unrestricted case, each circular list gives $n = b + g = 2g$ linear lists by cutting it arbitrarily. Thus we get $2(g!)^2/2g = g!\ (g-1)!$, which in this case is $3!\ 2! = 12$.

CL-2.2 Each of the 7 letters ABMNRST appears once and each of the letters CIO appears twice. Thus we must form a list of length k from the 10 distinct letters. The solutions are

$$\begin{aligned} k = 2&: & 10 \times 9 &= 90 \\ k = 3&: & 10 \times 9 \times 8 &= 720 \\ k = 4&: & 10 \times 9 \times 8 \times 7 &= 5040 \end{aligned}$$

CL-2.3 Each of the 7 letters ABMNRST appears once and each of the letters CIO appears twice.

- For $k = 2$, the letters are distinct OR equal. There are $(10)_2 = 90$ distinct choices. Since the only repeated letters are CIO, there are 3 ways to get equal letters. This gives 93.
- For $k = 3$, we have either all distinct ($(10)_3 = 720$) OR two equal. The two equal can be worked out as follows

  choose the repeated letter (3 ways) AND

  choose the positions for the two copies of the letter (3 ways) AND

  choose the remaining letter ($10 - 1 = 9$ ways).

  By the Rules of Sum and Product, we have $720 + 3 \times 9 \times 3 = 801$.

CL-2.4 (a) The letters are EILST. The number or 3-words is $(5)_3 = 60$.

(b) The answer is $5^3 = 125$.

(c) The letters are EILST, with T occurring 3-times, L occurring 2-times. Either the letters are distinct OR one letter appears twice OR one letter appears three times. We have seen that the first can be done in 60 ways. To do the second, choose one of L and T to repeat, choose one of the remaining 4 different letters and choose where that letter is to go, giving $2 \times 4 \times 3 = 24$. To do the third, use T. Thus, the answer is $60 + 24 + 1 = 85$.

CL-2.5 (a) Stripping off the initial R and terminal F, we are left with a list of at most 4 letters, at least one of which is an L. There is just 1 such list of length 1. There are $3^2 - 2^2 = 5$ lists of length 2, namely all those made from E, I and L minus those made from just

E and I. Similarly, there are $3^3 - 2^3 = 19$ of length 3 and $3^4 - 2^4 = 65$. This gives us a total of 90.

(b) The letters used are E, F, I, L and R in alphabetical order. To get the word before RELIEF, note that we cannot change just the F and/or the E to produce an earlier word. Thus we must change the I to get the preceding word. The first candidate in alphabetical order is F, giving us RELF. Working backwards in this manner, we come to RELELF, RELEIF, RELEF and, finally, RELEEF.

CL-2.6 (a) If there are 4 letters besides R and F, then there is only one R and one F, for a total of 65 spellings by the previous problem. If there are 3 letters besides R and F, we may have R$\cdots$F, R$\cdots$FF or RR$\cdots$F, which gives us $3 \times 19 = 57$ words by the previous problem. We'll say there are 3 RF patterns, namely RF, RFF and RRF. If there 2 letters besides R and F, there are 6 RF patterns, namely the three just listed, RFFF, RRFF and RRRF. This gives us $6 \times 5 = 30$ words. Finally, the last case has the 6 RF patterns just listed as well as RFFFF, RRFFF, RRRFF and RRRRF for a total of 10 patterns. This give us 10 words since the one remaining letter must be L. Adding up all these cases gives us $65 + 57 + 30 + 10 = 162$ possible spellings. Incidentally, there is a simple formula for the number of n long RF patterns, namely $n - 1$. Thus there are

$$1 + 2 + \ldots + (n-1) = n(n-1)/2$$

of length at most n. This gives our previous counts of 1, 3, 6 and 10.

(b) Reading toward the front of the dictionary from RELIEF we have RELIEF, RELFFF, RELFF, RELF, RELELF, RELEIF, RELEFF,..., and so the spelling five before RELIEF is RELEIF.

CL-2.7 There are $n!/(n-k)!$ lists of length k. The total number of lists (not counting the empty list) is

$$\begin{aligned} &\frac{n!}{(n-1)!} + \frac{n!}{(n-2)!} + \cdots + \frac{n!}{1!} + \frac{n!}{0!} \\ &= n!\left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{(n-1)!}\right) \\ &= n!\sum_{i=0}^{n-1} \frac{1^i}{i!}. \end{aligned}$$

Since $e = e^1 = \sum_{i=0}^{\infty} 1^i/i!$, it follows that the above sum is close to e.

CL-3.1 Choose values for pairs
AND choose suits for the lowest value pair
AND choose suits for the middle value pair
AND choose suits for the highest value pair.
This gives $\binom{13}{3}\binom{4}{2}^3 = 61,776$.

CL-3.2 Choose the lowest value in the straight (A to 10) AND choose a suit for each of the 5 values in the straight. This gives $10 \times 4^5 = 10240$.

Although the previous answer is acceptable, a poker player may object since a "straight flush" is better than a straight — and we included straight flushes in our count. Since a straight flush is a straight all in the same suit, we only have 4 choices of suits for the cards instead of 4^5. Thus, there are $10 \times 4 = 40$ straight flushes. Hence, the number of straights which are not straight flushes is $10240 - 40 = 10200$.

CL-3.3 If there are n 1's in the sequence, there are $n-1$ spaces between the 1's. Thus, there are 2^{n-1} compositions of n. A composition of n with k parts has $k-1$ commas The number of ways to insert $k-1$ commas into $n-1$ positions is $\binom{n-1}{k-1}$.

CL-3.4 Note that EXERCISES contains 3 E's, 2 S's and 1 each of C, I, R and X. We can use the multinomial coefficient

$$\binom{n}{m_1, m_2, \ldots, m_k} = \frac{n!}{m_1!\, m_2! \cdots m_k!}$$

where $n = m_1 + m_2 + \ldots + m_k$. Take $n = 9$, $m_1 = 3$, $m_2 = 2$ and $m_3 = m_4 = m_5 = m_6 = 1$. This gives $9!/3!\,2! = 30240$. This calculation can also be done without the use of a multinomial coefficient as follows. Choose 3 of the 9 possible positions to use for the three E's AND choose 2 of the 6 remaining positions to use for the two S's AND put a permutation of the remaining 4 letters in the remaining 4 places. This gives us $\binom{9}{3} \times \binom{6}{2} \times 4!$.

CL-3.5 An arrangement is a list formed from 13 things each used 4 times. Thus we have $n = 52$ and $m_i = 4$ for $1 \le i \le 13$ in the multinomial coefficient

$$\binom{n}{m_1, m_2, \ldots, m_k} = \frac{n!}{m_1!\, m_2! \cdots m_k!}.$$

CL-3.6 (a) The first 4 names in dictionary order are LALALAL, LALALAS, LALALAT, LALALIL.

(b) The last 4 names in dictionary order are TSITSAT, TSITSIL, TSITSIS, TSITSIT.

(c) To compute the names, we first find the possible consonant vowel patterns. They are CCVCCVC, CCVCVCC, CVCCVCC and CVCVCVC. The first three each contain two pairs of adjacent consonants, one isolated consonant and two vowels. Thus each corresponds to $(3 \times 2)^2 \times 3 \times 2^2$ names. The last has four isolated consonants and three vowels and so corresponds to $3^4 \times 2^3$ names. In total, there are 1,944 names.

CL-3.7 The first identity can be proved by writing the binomial coefficients in terms of factorials. It can also be proved from the definition of the binomial coefficient: Choosing a set of size k from a set of size n is equivalent to choosing a set of size $n-k$ to throw away, namely the things not chosen.

The total number of subsets of an n element set is 2^n. On the other hand, we can divide the subsets into collections T_j, where T_i contains all the i element subsets. The number of subsets in T_i is $\binom{n}{i}$. Apply the Rule of Sum.

CL-3.8 $S(n,n) = 1$: The only way to partition an n element set into n blocks is to put each element in a block by itself, so $S(n,n) = 1$. The only way to partition an n element set into one block is to put all the elements in the block, so $S(n,1) = 1$.

$S(n,n-1) = \binom{n}{2}$: The only way to partition an n element set into $n-1$ blocks is to choose two elements to be in a block together and put the remaining $n-2$ elements in $n-2$ blocks by themselves. Thus it suffices to choose the 2 elements that appear in a block together and so $S(n,n-1) = \binom{n}{2}$.

$S(n,1) = 1$: The only way to partition a set into one block is to put the entire set into the block.

$S(n,2) = (2^n-2)/2$: Note that $S(n,k)$ is the number of k-sets $\mathcal{S}$ where the entries in $\mathcal{S}$ are nonempty subsets of a given n-set T and each element of T appears in exactly one entry of $\mathcal{S}$. We will count k-lists, which is $k!$ times the number of k-sets. We choose a subset for the first block (first list entry) and use the remaining set elements for the second block. Since an n-set has 2^n, this would seem to give $2^n/2$; however, we must avoid empty blocks. In the ordered case, there are two ways this could happen since either the first or second list entry could be the empty set. Thus, we must have 2^n-2 instead of 2^n. The answer is $(2^n-2)/2$.

Here is another way to compute $S(n,2)$. Look at the block containing n. Once it is determined, the entire two block partition is determined. The block containing n can be gotten by starting with n and adjoining one of the $2^{n-1}-1$ proper subsets of $\{1,2,\ldots,n-1\}$.

CL-3.9 We use the hint. Choose i elements of $\{1,2,\cdots,n\}$ to be in the block with $n+1$ AND either do nothing else if $i=n$ OR partition the remaining elements. This gives $\binom{n}{n}$ if $i=n$ and $\binom{n}{i}B_{n-i}$ otherwise. If we set $B_0=1$, the second formula applies for $i=n$, too. Since $i=0$ OR $i=1$ OR $\cdots$ OR $i=n$, the result follows.

(b) To calculate B_n for $n \le 5$: We have $B_0 = 1$ from (a). Using the formula in (a) for $n=0,1,2,3,4$ in order, we obtain $B_1=1$, $B_2=2$, $B_3=5$, $B_4=15$ and $B_5=52$.

CL-3.10 (a) There is exactly one arrangement — 1,2,3,4,5,6,7,8,9.

(b) We do this by counting those arrangements that have $a_i \le a_{i+1}$ except, perhaps, for $i=5$. Then we subtract off those that also have $a_5 < a_6$. In set terms:

- S is the set of rearrangments for which $a_1 < a_2 < a_3 < a_4 < a_5$ and $a_6 < a_7 < a_8 < a_9$,
- T is the set of rearrangments for which $a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$, and
- we want $|S \setminus T| = |S| - |T|$.

An arrangement in S is completely determined by specifying the set $\{a_1,\ldots,a_5\}$, of which there are $\binom{9}{5} = 126$. In (a), we saw that $|T| = 1$. Thus the answer is $126-1=125$.

CL-4.1 Let the probability space consist of all $\binom{6}{2} = 15$ pairs of horses and use the uniform probability. Thus each pair has probability 1/15. Since each horse is in exactly 5 pairs, the probability of your choosing the winner is $5/15 = 1/3$, regardless of which horse wins.

Here is another way. You could choose your first horse and your second horse, so the space consists of 6×5 choices. The probability that your first choice was the winner is 1/6. The probability that your second choice was the winner is also 1/6. Since these events are disjoint, the probability of picking the winner is $1/6+1/6=1/3$.

Usually the probability of winning a bet on a horse race depends on picking the fastest horse after much study. The answer to this problem, 1/3, doesn't seem to have anything to do with studying the horses? Why?

CL-4.2 The sample space is $\{0, 1, \ldots, 36, 00\}$. We have $P(0) = P(1) = \cdots = P(36)$ and $P(00) = 1.05P(0)$. Thus

$$1 = P(0) + \cdots + P(36) + P(00) = 38.05P(0).$$

Hence $P(0) = 1/38.05$ and so $P(00) = 1.05/38.05 = 0.0276$.

CL-4.3 Let the event space be $\{A, B\}$, depending on who finds the key. Since Alice searches 20% faster than Bob, it is reasonable to assume that $P(A) = 1.2\,P(B)$. The odds that Alice finds the key are $P(A)/P(B) = 1.2$, that is, 1.2:1, which can also be written as 6:5. Combining $P(A) = 1.2\,P(B)$ with $P(A) + P(B) = 1$, we find that $P(A) = 1.2/2.2 = 0.545$.

CL-4.4 Let A be the event that you pick the winner and B the probability that you pick the horse that places. From a previous exercise, $P(A) = 1/3$ Similarly, $P(B) = 1/3$. We want $P(A \cup B)$. By the principle of inclusion and exclusion, this is $P(A) + P(B) - P(A \cap B)$. Of all $\binom{6}{2} = 15$ choices, only one is in $A \cap B$. Thus $P(A \cap B) = 1/15$ and the answer is $1/3 + 1/3 - 1/15 = 3/5$.

CL-4.5 Since probabilities are uniform, we simply count the number of events that satisfy the conditions and divide by the total number of events, which is m^n for n balls and m boxes. First will do the problems in an ad hoc manner, then we'll discuss a systematic solution. We use (a')–(c') to denote the answers for (d).

(a) We place one ball in the first box AND one in the second AND so on. Since this can be done in 4! ways, the answer is $4!/4^4 = 3/32$.

(a') We must have one box with two balls and one ball in each of the other three boxes. We choose one box to contain two balls AND two balls for the box AND distribute the three remaining balls into three boxes as in (a). This gives us $4 \times \binom{5}{2} \times 3! = 240$. Thus the answer is $240/4^5 = 15/64$.

(b) This is somewhat like (a'). Choose a box to be empty AND choose a box to contain two balls AND choose two balls for the box AND distribute the other two balls into the other two boxes. This gives $4 \times 3 \times \binom{4}{2} \times 2! = 144$. Thus the answer is $144/4^4 = 9/16$.

(b') This is more complicated since the ball counts can be either 3,1,1,0 or 2,2,1,0. As in (b), there are $4 \times 3 \times \binom{5}{3} \times 2! = 240$ to do the first. In the second, there are $\binom{4}{2} \times 2 = 12$ ways to designate the boxes and $\binom{5}{2} \times \binom{3}{2} = 30$ ways to choose the balls for the boxes that contain two each. Thus there are 360 ways and the answer is $(240 + 360)/4^5 = 75/128$.

(c) Simply subtract the answer for (a) from 1 since we are asking for the complementary event. This gives 29/32. For (c') we have 39/64.

We now consider a systematic approach. Suppose we want to assign n balls to m boxes so that exactly $k \le m$ of the boxes contain balls. Call the balls $1, 2, \ldots, n$ First partition the set of n balls into k blocks. This can be done in $S(n, k)$ ways, where $S(n, k)$ is the Stirling number discussed in Section 3. List the blocks in some order (pick your favorite; e.g., numerical order based on the smallest element in the block). Assign the first block to a box AND assign the second block to a box AND, etc. This can be done in $m(m-1)\cdots(m-k+1) = m!/(m-k)!$ ways. Hence the the number of ways to distribute the balls is $S(n,k)m!/(m-k)!$ and so the probability is $S(n,k)m!/(m-k)!\,m^n$. For our particular problems, the answers are

(a)	$S(4,4)4!/0!\,4^4 = 3/32$	(a')	$S(5,4)4!/0!\,4^5 = 15/64$
(b)	$S(4,3)4!/1!\,4^4 = 9/16$	(b')	$S(5,3)4!/1!\,4^5 = 75/128\,.$

The moral here is that if you can think of a systematic approach to a class of problems, it is likely to be easier than solving each problem separately.

CL-4.6 (a) Since the die is thrown k times, the sample space is S^k, where $S = \{1,2,3,4,5,6\}$. Since the die is fair, all 6^k sequences in S^k are equally likely. We claim that exactly half have an even sum and so $P(E) = 1/2$. Why do half have an even sum? Here are two proofs.

- Let $N_o(n)$ be the number of odd sums in the first n throws and let $N_e(n)$ be the number of even sums. We have

  $$N_e(k) = 3N_e(k-1) + 3N_o(k-1) \qquad \text{and} \qquad N_o(k) = 3N_o(k-1) + 3N_e(k-1)$$

  because an even sum is obtained from an even by throwing 2, 4, or 6 and from an odd by throwing 1, 3, or 5; and similarly for an odd sum. Thus $N_e(k) = N_o(k)$. Since the probability on S^k is uniform, the probability of an even sum is 1/2.

- Let S_o be all the k-lists in S^k with odd sum and let S_e be those with even sum. Define the function $f : S^k \to S^k$ as follows

  $$f(x_1, x_2 \ldots, x_k) = \begin{cases} (x_1+1, x_2, \ldots, x_k), & \text{if } x_1 \text{ is odd;} \\ (x_1-1, x_2, \ldots, x_k), & \text{if } x_1 \text{ is even.} \end{cases}$$

  We leave it to you to convince yourself that this function is a bijection between S_o and S_e. (A bijection is a one-to-one correspondence between elements of S_o and S_e.)

RELATED PROBLEMS TO THINK ABOUT: Suppose that p_1, p_2, p_3, p_4, p_5, and p_6 represent the probabilities that a face with number 1, 2, 3, 4, 5, or 6 is rolled (appears on the top face of the die). What is the probability that a sequence of 3 throws of the die produces an even sum? What is the probability that 4 throws produce an even sum? What is the probability that k throws produce an even sum?
Hint: Let $P_e = p_2+p_4+p_6$ be the probability of an even number on a single throw, $P_o = 1 - P_e$ the probability of an odd number. Compare terms in $(P_o+P_e)^k$ and $(P_o-P_e)^k$ by using the binomial theorem. To do this, you will need to know that probabilities multiply for independent events. (See the index for "independent events.")

(b) The sample space for drawing cards n times is S^n where S is the Cartesian product

$$\{\mathrm{A}, 2, 3, \ldots, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\} \times \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

The probability of any point in S^n is $(1/52)^n$. The number of draws with no king is $(52-4)^n$ and so the probability of none is $(48/52)^n = (12/13)^n$. The probability of at least one king is $1-(12/13)^n$.

(c) The equiprobable sample space is gotten by distinguishing the marbles $M = \{w_1, w_2, w_3, r_1, \ldots\}$ and defining the sample space by

$$S = \{(m, m') : m \text{ and } m' \text{ are distinct elements of } M\}.$$

If E_r is the event that both m and m' are red, then $P(E_r) = 4{*}3/|S|$ where $|S| = 13{*}12$.

RELATED PROBLEMS TO THINK ABOUT: What is the probability of two white and two blue marbles being drawn if four marbles are drawn without replacement? Of two white and two blue marbles being drawn if four marbles are drawn with replacement?

CL-4.7 This is nearly identical to the example on hypergeometric probabilities. The answer is $C(5,3)C(10,3)/C(15,6)$.

CL-4.8 Let $B = \{1, 2, \ldots, 10\}$.

(a) The sample space S is the set of all subsets of B of size 2. Thus $|S| = \binom{10}{2} = 45$. Since each draw is equally likely, we just need to know how many pairs have an odd sum. One of the balls must have an odd label and the other an even label. The number of pairs with this property is 5×5 since there are 5 odd labels and 5 even labels. Thus the probability is $25/45 = 5/9$.

(b) The sample space S is the set of ordered pairs (b_1, b_2) with $b_1 \neq b_2$ both from B. Thus $|S| = 10 \times 9 = 90$. To get an odd sum, one of b_1 and b_2 must be even and the other odd. Thus there are 10 choices for b_1 AND then 5 choices for b_2. The probability is $50/90 = 5/9$.

(c) The sample space is $S = B \times B$ and $|S| = 100$. The number of pairs (b_1, b_2) is 50 as in (b). Thus the probability is $50/100 = 1/2$.

CL-4.9 This is an inclusion and exclusion type of problem. There are three ways to approach such problems:

- Have a variety of formulas handy that you can plug into. This, by itself, is not a good idea because you may encounter a problem that doesn't fit any of the formulas you know.
- Draw a Venn diagram and use the information you have to compute the probability of as many regions as you can. If there are more than 3 sets, the Venn diagram is too confusing to be very useful. With 2 or 3 sets, it is a good approach.
- Carry out the preceding idea without the picture. We do this here.

Suppose we are dealing with k sets, $A_1, \ldots, A_k$. We need to know what the regions in the Venn diagram are. Each region corresponds to $T_1 \cap \cdots \cap T_k$ where T_i is either A_i or A_i^c. In our case, $k = 2$ and so the probabilities of the regions are

$$P(A \cap B) \qquad P(A \cap B^c) \qquad P(A^c \cap B) \qquad P(A^c \cap B^c).$$

We get A by combining $A \cap B$ and $A \cap B^c$. We get B by combining $A \cap B$ and $A^c \cap B$. By properties of sets, $(A \cup B)^c = A^c \cap B^c$. Thus our data corresponds to the three equations

$$P(A \cap B) + P(A \cap B^c) = 3/8 \qquad P(A \cap B) + P(A^c \cap B) = 1/2 \qquad P(A^c \cap B^c) = 3/8.$$

We have one other equation: The probabilities of all four regions sum to 1. This gives us four equations in four unknowns whose solution is

$$P(A \cap B) = 1/4 \qquad P(A \cap B^c) = 1/8 \qquad P(A^c \cap B) = 1/4 \qquad P(A^c \cap B^c) = 3/8.$$

Thus the answer to the problem is $1/4$.

When we are not asked for the probability of all regions, it is often possible to take shortcuts. That is the case here. From $P((A \cup B)^c) = 3/8$ we have $P(A \cup B) = 1 - 3/8 = 5/8$. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and three of the four terms in this equation are known, we can easily solve for $P(A \cap B)$.

CL-4.10 This is another Venn diagram problem. This time we'll work with number of people instead of probabilities. Let C correspond to the set of computer science majors, W the set of women and S to the entire student body. We are given

$$\begin{aligned} |C| &= 20\% \times 5,000 = 1,000 \\ |W| &= 58\% \times 5,000 = 2,900 \\ |C \cap W| &= 430. \end{aligned}$$

(a) We want $|W \cap C^c|$, which equals $|W| - |W \cap C| = 2,470$. You should be able to see why this is so by the Venn diagram or by the method used in the previous problem.

(b) The number of men who are computer science majors is the number of computer science majors who are not women. This is $|C| - |C \cap W| = 1,000 - 430 = 570$. The number of men in the student body is $42\% \times 5,000 = 2,100$. Thus $2,100 - 570 = 1,530$ men are not computer science majors.

CL-4.11 Since the coin is fair $P(H) = 1/2$, what about $P(W)$, the probability that Beatlebomb wins? Recall the meaning of the English phrase "the odds that it will occur." This is trivial but important, as the phrase is used often in everyday applications of probability. If you don't recall the meaning, see the discussion of odds in the text. From the definition of odds, you should be able to show that $P(W) = 1/101$. If we had studied "independent" events, you could immediately see that the answer to the questions is $(1/2) \times (1/101) = 1/202$, but we need a different approach which lets independent events sneak in through the back door.

Let the sample space be $\{H, T\} \times \{W, L\}$, corresponding to the outcome of the coin toss and the outcome of the race. From the previous paragraph $P(\{(H, W), (T, W)\}) = 1/101$. Since the coin is fair and the coin toss doesn't influence the race, we should have $P((H, W)) = P((T, W))$. Since

$$P(\{(H, W), (T, W)\}) = P((H, W)) + P((T, W)),$$

It follows after a little algebra that $P(H, W)) = 1/202$.

CL-4.12 This is another example of the hypergeometric probabililty. Do you see why? The answer is $C(37, 11)C(2, 2)/C(39, 13)$.

CL-4.13 (a) Let words of length 6 formed from three G's and three B's stand for the arrangements in terms of Boys and Girls; for example, BBGGBG or BBBGGG. There are $\binom{6}{3} = 6!/(3!\,3!) = 20$ such words. Four such words correspond to the three girls together: GGGBBB, BGGGBB, BBGGGB, BBBGGG. The probability of three girls being together is $4/20 = 1/5$.

(b) If they are then seated around a circular table, there are two additional arrangements that will result in all three girls sitting together: GGBBBG and GBBBGG. The probability is $6/20 = 3/10$.

CL-4.14 You can draw the Venn diagram for three sets and, for each of the eight regions, count how much a point in the region contributes to the addition and subtraction. This does not extend to the general case. We give another proof that does.

Let S be the sample space and let T be a subset of S Define the function χ_T with domain S by

$$\chi_T(s) = \begin{cases} 1 & \text{if } s \in T, \\ 0 & \text{if } s \notin T. \end{cases}$$

This is called the *characteristic function* of T.[1] We leave it to you to check that

$$\chi_{T^c}(s) = 1 - \chi_T(s), \quad \chi_{T\cap U}(s) = \chi_T(s)\,\chi_U(s), \quad \text{and} \quad P(S) = \sum_{s\in S} P(s)\chi_T(s).$$

Using these equations and a little algebra, we have

$$\begin{aligned} P(A^c \cap B^c \cap C^c) &= \sum_{s\in S} P(s)\chi_{A^c\cap B^c\cap C^c}(s) \\ &= \sum_{s\in S} P(s)\big(1-\chi_A(s)\big)\big(1-\chi_B(s)\big)\big(1-\chi_C(s)\big) \\ &= \sum_{s\in S} P(s) - \sum_{s\in S} P(s)\chi_A(s) - \sum_{s\in S} P(s)\chi_B(s) - \sum_{s\in S} P(s)\chi_C(s) \\ &\quad + \sum_{s\in S} P(s)\chi_A(s)\chi_B(s) + \sum_{s\in S} P(s)\chi_A(s)\chi_C(s) \\ &\quad + \sum_{s\in S} P(s)\chi_B(s)\chi_C(s) - \sum_{s\in S} P(s)\chi_A(s)\chi_B(s)\chi_C(s) \\ &= 1 - P(A) - P(B) - P(C) \\ &\quad + P(A\cap B) + P(A\cap C) \\ &\quad + P(B\cap C) - P(A\cap B\cap C). \end{aligned}$$

CL-4.15 Let the stick have unit length and let x be the distance from the end of the stick where the break is made. Thus $0 \le x \le 1$. The longer piece will be at least twice the length of the shorter if $x \le 1/3$ or if $x \ge 2/3$. The probability of this is $1/3 + 1/3 = 2/3$. You should be able to fill in the details.

CL-4.16 Let x and y be the places where the stick is broken. Thus, (x, y) is chosen uniformly at random in the square $S = (0, 1) \times (0, 1)$. Three pieces form a triangle if the sum of the lengths of any two is always greater than the length of the third. We must determine which regions in S satisfy this condition.

Suppose $x < y$. The lengths are then x, $y - x$, and $1 - y$. The conditions are

$$x + (y-x) > 1-y, \quad x + (1-y) > y-x, \quad \text{and} \quad (y-x) + (1-y) > x.$$

With a little algebra, these become

$$y > 1/2, \quad y < x + 1/2, \quad \text{and} \quad x < 1/2,$$

[1] χ is a lower case Greek letter and is pronounced like the "ki" in "kind."

respectively. If you draw a picture, you will see that this is a triangle of area $1/8$.

If $x > y$, we obtain the same results with the roles of x and y reversed. Thus the total area is $1/8 + 1/8 = 1/4$. Since S has area 1, the probability is $1/4$.

CL-4.17 Look where the center of the coin lands. If it is within $d/2$ of a lattice point, it covers the lattice point. Thus, there is a circle of diameter d about each lattice point and the coin covers a lattice point if and only if it lands in one of the circles. We need to compute the fraction of the plane covered by these circles. Since the pattern repeats in a regular fashion, all we need to do is calculate the fraction of the square $\{(x, y)|0 \le x \le 1, 0 \le y \le 1\}$ that contains parts of circles. There is a quarter circle about each of the points (0,0), (0,1), (1,0) and (1,1) inside the square. Since the circle has diameter at most 1, the quarter circles have no area in common and so their total area equals the area of the coin, $\pi d^2/4$. Since the are of the square is 1, the probability that the coin covers a lattice point is $\pi d^2/4$.

CL-4.18 Select the three points uniformly at random from the circumference of the circle and label them 1, 2, 3 going clockwise around the circle from the top of the circle. Let E_1 denote the event consisting of all such configurations where points 2 and 3 lie in the half circle starting at 1 and going clockwise (180 degrees). Let E_2 denote the event that points 2 and 1 lie in the half circle starting at 2 and going clockwise 180 degrees. Let E_3 be defined similarly. Note that the events E_1, E_2, and E_3 are mutually exclusive. (Draw a picture and think about this.) By our basic probability axioms, the probability of the union is the sum of the probabilities $P(E_1) + P(E_2) + P(E_3)$. To compute $P(E_1)$, imagine point 1 on the circle, consider its associated half circle and, before looking at the other two points, ask "What is the probability that they lie in that half circle?" Let x be the number of degrees clockwise from point 1 to point 2 and y the number from 1 to 3. Thus (x, y) is a point chosen uniformly at random in the square $[0, 360) \times [0, 360)$. For event E_1 to occur, (x, y) must lie in $[0, 180) \times [0, 180)$, which is 1/4 of the original square. Thus $P(E_1) = 1/4$. (This can also be done using independent events: the locations of points 2 and 3 are chosen independently so one gets $(1/2) \times (1/2)$.) The probabilities of E_2 and E_3 are the same for the same reason. Thus $P(E_1) + P(E_2) + P(E_3) = 3/4$.

What is the probability that k points selected uniformly at random on the circumference of a circle lie the same semicircle? Use the same method. The answer is $k/(2^{k-1})$.