Solutions for Decision Trees and Recursion

DT-1.1 PREV: C, CC, CCV, CCVC, CCVCC, CCCVCV, CV, CVC, CVCC, CVCCV, CVCV, CVCVC, V, VC, VCC, VCCV, VCCVC, VCV, VCVC, VCVCC, VCVCV.
POSV: CCVCC, CCVCV, CCVC, CCV, CC, CVCCV, CVCC, CVCVC, CVCV, CVC, CV, C, VCCVC, VCCV, VCC, VCVCC, VCVCV, VCVC, VCV, VC, V.
BFV: C, V, CC, CV, VC, CCV, CVC, VCC, VCV, CCVC, CVCC, CVCV, VCCV, VCVC, CCVCC, CCVCV, CVCCV, CVCVC, VCCVC, VCVCC, VCVCV.

DT-1.2 You will need the decision trees for lex and insertion order for permutations of $\underline{3}$ and $\underline{4}$. The text gives the tree for insertion order for $\underline{4}$, from which the tree for $\underline{3}$ can be found — just stop one level above the leaves of $\underline{4}$. You should construct the tree for lex order.

(a) To answer this, compare the leaves. For $n = 3$, permutations $\sigma = 123$, 132, and 321 have $\text{RANK}_L(\sigma) = \text{RANK}_I(\sigma)$. For $n = 4$ the permutations $\sigma = 1234$, 1243, and 4321 have $\text{RANK}_L(\sigma) = \text{RANK}_I(\sigma)$.

(b) From the tree for (a), $\text{RANK}_L(2314) = 8$.
Rather than draw the large tree for $\underline{5}$, we use a smarter approach to compute $\text{RANK}_L(45321) = 95$. To see the latter, Note that all permutations on $\underline{5}$ that start with 1, 2, or 3 come before 45321. There are $3 \times 24 = 72$ of those. This leads us to the subtree for permutations of $\{1, 2, 3, 5\}$ in lex order. It looks just like the decision tree for $\underline{4}$ with 4 replaced by 5. (Why is this?) Since $\text{RANK}_L(4321) = 23$, this makes a total of $72 + 23 = 95$ permutations that come before 45321 and so $\text{RANK}_L(45321) = 95$. If you find this unclear, you should try to draw a picture to help you understand it.

(c) $\text{RANK}_I(2314) = 16$. What about $\text{RANK}_I(45321)$? First does 1, then 2, and so on. After have done all but 5, we are at the rightmost leaf of the tree for $\underline{4}$. It has 23 leaves to the left of it. When we insert 5, each of these leaves is replaced by 5 new leaves because there are 5 places to insert 5. This gives us $5 \times 23 = 115$ leaves. Finally, of the 5 places we could insert 5 into 4321, we chose the 4th so there are 3 additional leaves to the left of it. Thus the rank is $115 = 3 = 118$.

(d) $\text{RANK}_L(3241) = 15$.

(e) $\text{RANK}_I(4213) = 15$.

(f) The first 24 permutations on $\underline{5}$ consist of 1 followed by a permutation on $\{2, 3, 4, 5\}$. Since our goal is the permutation of rank 15, it is in this set. By (d), RANK_L of 3241 is 15 for $n = 4$. Thus $\text{RANK}_L(4352) = 15$ in the lex list of permutations on $\{2, 3, 4, 5\}$.

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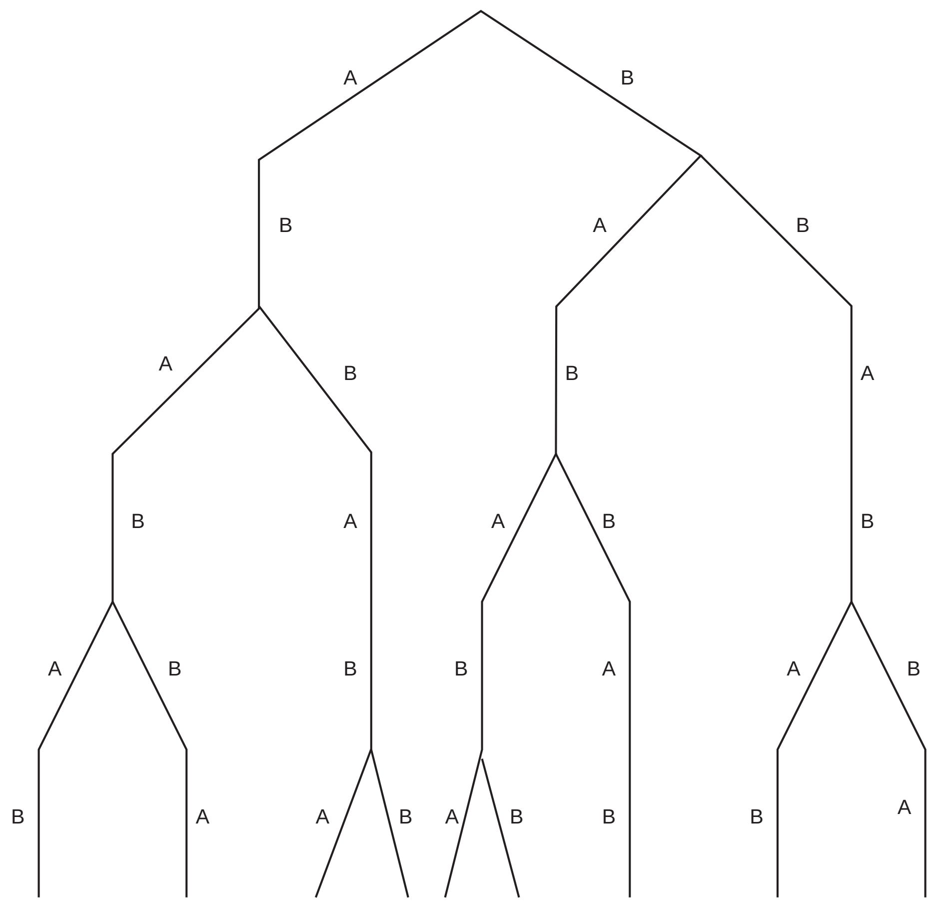
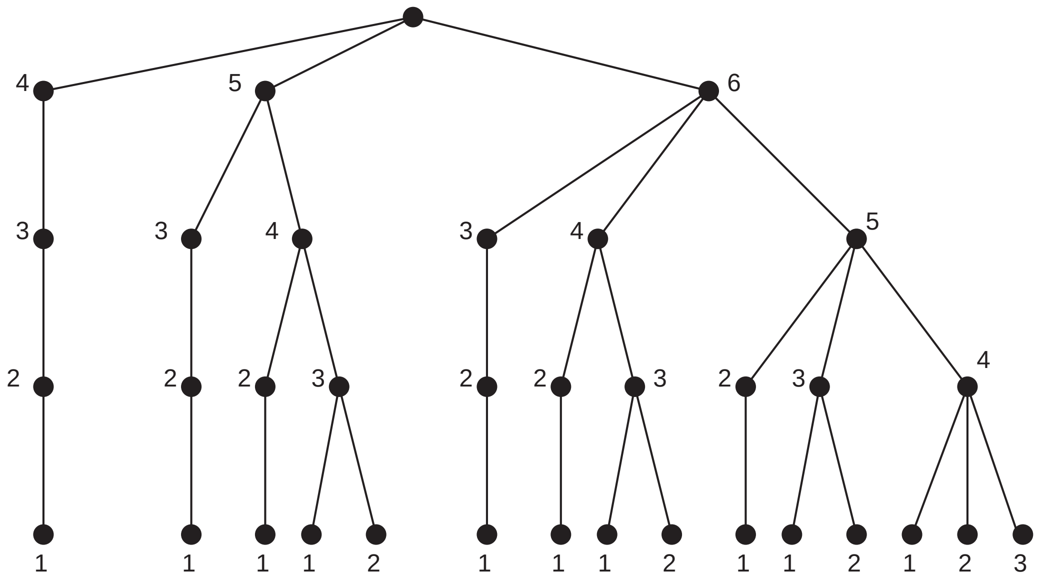
DT-1.3 Here is the tree



The list in lex order:

ABABAB ABABBA, ABBABA, ABBABB BABABA BABABB BABBAB BBABAB BBABBA

DT-1.4 Here is a decision tree for $D(\underline{6}^4)$. The leaves correspond to the elements of $D(\underline{6}^4)$ in lex order, obtained by reading the sequence of vertex labels from the root to the leaf.



(a) The rank of 5431 is 3. The rank of 6531 is 10.

(b) 4321 has rank 0 and 6431 has rank 7.

(c) The first 5 leaves correspond to $D(\underline{5}^{\underline{4}})$.

(d) $D(\underline{6}^{\underline{4}})$ is bijectively equivalent to the set, $\mathbf{P}(\underline{6}, 4)$, of all subsets of $\underline{6}$ of size 4. Under this bijection, an element such as $5431 \in D(\underline{6}^{\underline{4}})$ corresponds to the set $\{1, 3, 4, 5\}$.

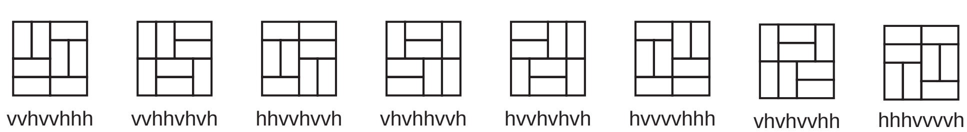
DT-1.5 For PREV and POSV, omit Step 2. For PREV, begin Step 3 with the sentence

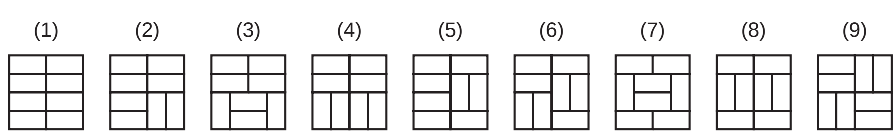
> "If you have not used any edges leading out from the vertex, list the vertex."

For POSV, change Step 3 to

> "If there are no unused edges leading out from the vertex, list the vertex
> and go to Step 4; otherwise, go to Step 5."

DT-1.6 The problem is that the eight hibachi grills, though different as domino coverings, are all equivalent or "isomorphic" once they are made into grills. All eight in the first row below can be gotten by rotating and/or turning over the first grill.





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There are nine different grills as shown in the picture. These nine might be called a "representative system" for the domino coverings up to "grill equivalence." Note that these nine representatives are listed in lex order according to their codes (starting with hhhhhhhh and ending with hvvhvvhh). They each have another interesting property: each one is lexicographically minimal among all patterns equivalent to it. The one we selected from the list of "screwup" grills (number (6)) has code hhhvvvvh and that is minimal among all codes on the first row of coverings.

This problem is representative of an important class of problems called "isomorph rejection problems." The technique we have illustrated, selecting a lex minimal system of representatives up to some sort of equivalence relation, is an important technique in this subject.

DT-2.1 (a) Let $\mathcal{A}(n)$ be the assertion that $G(n) = (1-A^n)/(1-A)$. When $n=1$, $G(1)=1$ and $(1-A^n)/(1-A)=1$, so the base case is proved. For $n>1$, we have

$$\begin{aligned} G(n) &= 1 + A + A^2 + \ldots + A^{n-1} && \text{by definition,}\\ &= (1 + A + A^2 + \ldots + A^{n-2}) + A^{n-1} \\ &= \frac{1-A^{n-1}}{1-A} + A^{n-1} && \text{by } \mathcal{A}(n-1),\\ &= \frac{1-A^n}{1-A} && \text{by algebra.} \end{aligned}$$

(b) The recursion can be found by looking at the definition or by examining the proof in (a). It is $G(1)=1$ and, for $n>1$, $G(n) = G(n-1) + A^{n-1}$.

(c) Applying the theorem is straightforward. The formula equals 1 when $n=1$, which agrees with $G(1)$. By some simple algebra

$$\frac{1-A^{n-1}}{1-A} + A^{n-1} = \frac{(1-A^{n-1}) + (A^{n-1}-A^n)}{1-A} = \frac{1-A^n}{1-A},$$

and so the formula satisfies the recursion.

(d) Letting $A = y/x$ and cleaning up some fractions

$$\frac{1-(y/x)^n}{1-y/x} = \frac{y^n - x^n}{x-y}\, x^{n-1}.$$

Let $n = k+1$, multiply by x^k and use the geometric series to obtain

$$\begin{aligned} \frac{x^{k+1}D - y^{k+1}}{x-y} &= x^k\Big(1 + (y/x) + (y/x)^2 + \cdots + (y/x)^k\Big)\\ &= x^k y^0 + x^{k-1}y^1 + \cdots + x^0 y^k. \end{aligned}$$

DT-2.2 We will Theorem 3 to prove our conjectures are correct.

(a) Writing out the first few terms gives A, $A/(1+A)$, $A/(1+2A)$, $A/(1+3A)$, etc. It appears that $a_k = A/(1+kA)$. Since $A>0$, the denominators are never zero.

When $k = 0$, $A/(1 + kA) = A$, which satisfies the initial condition. We check the recursion:

$$\frac{A/(1+(k-1)A)}{1+A/(1+(k-1)A)} = \frac{A}{(1+(k-1)A)+A} = A/(1+kA),$$

which is the conjectured value for a_k.

(b) Writing out the first few terms gives C, $AC + B$, $A^2C + AB + B$, $A^3C + A^2B + AB + B$, $A^4C + A^3B + A^2B + AB + B$, etc. Here is one possible formula:

$$a_k = A^kC + B(1 + A + A^2 + \ldots + A^{k-1}).$$

Here is a second possibility:

$$a_k = A^kC + B\left(\frac{1-A^k}{1-A}\right).$$

Using the previous exercise, you can see that they are equal. We leave it to you to give a proof of correctness for both formulas, without using the previous exercise.

DT-2.3 We use Theorem 3. The formula gives the correct value for $k = 0$. The recursion checks because

$$\begin{aligned} A + B(k-1)\Big(((k-1)^2-1)/3\Big) + Bk(k-1) &= A + B(k-1)\Big((k^2-2k+1-1)-3k\Big) \\ &= A + B(k-1)k(k+1)/3 \\ &= A + Bk(k^2-1)/3. \end{aligned}$$

This completes the proof.

DT-2.4 (a) We apply Theorem 3, but there is a little complication: The formula starts at $k = 1$, so we cannot check the recursion for $k = 1$. Thus we need a_1 to be the initial condition. From the recursion, $a_1 = 2A - C$, which we take as our initial condition and use the recursion for $k > 1$. You should verify that the formula gives a_1 correctly and that the formula satisfies the recursion when $k > 1$.

(b) From the last part of Exercise 1 with $x = 2$ and $y = -1$, we obtain

$$a_k = A\left(\frac{2^{k+1} - (-1)^{k+1}}{3}\right) + (-1)^k(C - A).$$

Make sure you can do the calculations to derive this.

DT-2.5 The characteristic equation is $r^2 - 2r + 1 = (r-1)^2 = 0$. Thus, $r = s = 1$. So, we have the two sequences $0, s, 2s^2, 3s^3, 4s^4 \ldots$ and $1, s, s^2, s^3, \ldots$ as solutions. We need to satisfy the initial conditions. We have

$$\begin{array}{ccccc} \alpha & + & 0\beta & = & 2 \\ s\alpha & + & s\beta & = & 1 \end{array}$$

where $s = 1$. We easily obtain $\alpha = 2$ and $\beta = -1$. Thus, the sequence $2, 1, 0, -1, -2, -3, -4, \ldots$ is the required solution.

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DT-2.6 (a) The initial condition ($D_0 = 1$) is correct. We want to use induction to prove the recursion. The base case for the *induction proof* need not be the same as the base case for the *recursive equation*. In fact, it is not. We could simply tell you what it is, but it's better to discover it. The discovery comes when we attempt a proof and see what conditions we need on n. Let $\mathcal{A}(n)$ be "$D_n = D_{n-1} + (-1)^n$." We have

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2} \quad \text{for } n \geq 2 \tag{$*$}$$

and we want to somehow convert this to $D_n = nD_{n-1} + (-1)^n$. Thus we must replace D_{n-2} by either D_n or D_{n-1} using $\mathcal{A}(k)$ with $k < n$. Since $\mathcal{A}(n-1)$ involved D_{n-1} and D_{n-2}, we can solve it for the latter in terms of the former: $(n-1)D_{n-2} = D_{n-1} - (-1)^{n-1}$, provided $n - 1 \geq 1$. Combining this with (*) we have

$$D_n = (n-1)D_{n-1} + D_{n-1} - (-1)^{n-1} = nD_{n-1} + (-1)^n \quad \text{for } n \geq 2.$$

Thus $n = 1$ is the base case for the induction proof. It is easy to check that $D_1 = D_0 + (-1)^1$.

(b) The initial condition is for $n = 0$ and it is easily checked. For $n > 0$, it suffices to verify that the formula satisfies the recursion. Here it is

$$n\left((n-1)!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\right) + (-1)^n = n!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!} + n!\,\frac{(-1)^n}{n!} = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

(c) This is like (b). The initial conditions are easy. We check the recursion. With some algebra,

$$\begin{aligned}
&(n-1)\left((n-1)!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}(n-2)!\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}\right)\\
&\quad= (n-1)\left(\Big((n-1)! + (n-2)!\Big)\sum_{k=0}^{n-2}\frac{(-1)^k}{k!} + (n-1)!\,\frac{(-1)^{n-1}}{(n-1)!}\right)\\
&\quad= n!\sum_{k=0}^{n-2}\frac{(-1)^k}{k!} + (n-1)(-1)^{n-1}.
\end{aligned} \tag{$*$}$$

We need this to equal $n!\sum_{k=0}^{n}(-1)^k/k!$. Since this sum has two more terms than the sum in ($*$), we will have equality if and only if the last two terms equal $(n-1)(-1)^{n-1}$. That is, we need

$$(n-1)(-1)^{n-1} = n!\Big((-1)^{n-1}/(n-1)! + (-1)^n/n!\Big).$$

With some algebra, we can rearrange the right side as

$$(-1)^{n-1}n + (-1)^n = (-1)^{n-1}(n-1)$$

and so we are done.

DT-3.1 We refer to the decision tree in Example 17. The permutation 87612345 specifies, by edge labels, a path from the root L($\underline{8}$) to a leaf in the decision tree. To compute the rank, we must compute the number of leaves "abandoned" by each edge just as was done in Example 21. There are eight edges in the path with the number of abandoned leaves equal to $7\times7!+6\times6!+5\times5!+0+0+0+0+0 = 35,280+4,320+600 = 40,200$. This is the RANK of 87612345 in the lex list of permutations on $\underline{8}$. Note that $8! = 40,320$, so the RANK 20,160 permutation is the first one of the second half of the list: 51234678.

DT-3.2 (a) The corresponding path in the decision tree is H(8, S, E, G), H(7, E, S, G), H(6, S, E, G), H(5, S, G, E), H(4, S, E, G), H(3, E, S, G), E $\xrightarrow{3}$ G.

(b) The move that produced the configuration of (a) was E $\xrightarrow{3}$ G. The configuration prior to that was Pole S: 6, 5, 2, 1; Pole E: 3; Pole G: 8, 7, 4.

(c) The move just prior to E $\xrightarrow{3}$ G was G $\xrightarrow{1}$ S. This is seen from the decision tree structure or from the fact that the smallest washer, number 1, moves every other time in the pattern S, E, G, S, E, G, etc. The configuration just prior to the move G $\xrightarrow{1}$ S was Pole S: 6, 5, 2; Pole E: 3; Pole G: 8, 7, 4, 1.

(d) The next move after E $\xrightarrow{3}$ G will be another move by washer 1 in its tiresome cycle S, E, G, S, E, G, etc. That will be S $\xrightarrow{1}$ E.

(e) The RANK of the move that produced (a) can be computed by summing the abandoned leaves associated with each edge of the path (a) in the decision tree. (See Example 21.) There are six edges in the path of part (a) with associated abandoned leaves being $2^7 = 128$, $2^6 = 64$, 0, 0, $2^3 = 8$, $2^2 - 1 = 3$. The total is 203.

DT-3.3 (a) 110010000 is followed by 110110000 and preceded by 110010001.

(b) The first element of the second half of the list corresponds to a path in the decision tree that starts with a right-sloping edge and has all of the remaining eight edges left-sloping. That element is 110000000.

(c) Each right-sloping edge abandons 2^{n-k} leaves, if the edge is the k^{th} one in the path. For the path 111111111 the right-sloping edges are numbers 1, 3, 5, 7, and 9 (remember, after the first edge, a label 1 causes the direction of the path to change). Thus, the rank of 111111111 is $2^8 + 2^6 + 2^4 + 2^2 + 2^0 = 341$.

(d) To compute the element of RANK 372, we first compute the path in the decision tree that corresponds to the element. The first edge must be **(1) right sloping** (abandoning 256 leaves), since the largest rank of any leaf at the end of a path that starts left sloping is $2^8 - 1 = 255$. We apply this same reasoning recursively. The right sloping edge leads to 256 leaves. We wish to find the leaf of RANK $372 - 256 = 116$ in that list of 256 leaves. That means the second edge must be **(1) left sloping** (abandoning 0 leaves), so our path starts off **(1) right sloping**, **(1) left sloping**. This path can access 128 leaves. We want the leaf of RANK $116 - 0$ in this list. Thus we must access a leaf in the second half of the list of 128, so the third edge must be **(1) right sloping** (abandoning 64 leaves). In that second half we must find the leaf of RANK $116 - 64 = 52$.

Our path is now **(1) right sloping**, **(1) left sloping**, **(1) right sloping**. Following that path leads to 64 leaves of which we want the leaf of RANK 52. Thus, the fourth edge must be **(0) right sloping** (abandoning 32 leaves). This path of four edges leads to 32 leaves of which we must find the one of RANK $52 - 32 = 20$. Thus the fifth edge must also be **(0) right sloping** (abandoning 16 leaves). Thus we must find the leaf of RANK $20 - 16 = 4$. This means that the sixth edge must be **(1) left sloping** (abandoning 0 leaves), the seventh edge must be **(1) right sloping** (abandoning 4 leaves), and the last two edges must be left sloping: **(1) left sloping** (abandoning 0 leaves), **(0) left sloping** (abandoning 0 leaves). Thus the final path is 111001110.

DT-3.4 (a) Let $\mathcal{A}(n)$ be the assertion "H(n,S,E,G) takes the least number of moves." Clearly $\mathcal{A}(1)$ is true since only one move is required. We now prove $\mathcal{A}(n)$. Note that to do $\mathrm{S} \xrightarrow{n} \mathrm{G}$ we must first move all the other washers to pole E. They can be stacked only one way on pole E, so moving the washers from S to E requires using a solution to the Towers of Hanoi problem for $n-1$ washers. By $\mathcal{A}(n-1)$, this is done in the least number of moves by H($n-1$,S,G,E). Similarly, H($n-1$,E,S,G) moves these washers to G in the least number of moves.

(b) For $n = 1$, $f_1 = 1$: $\mathrm{S} \xrightarrow{1} \mathrm{G}$
For $n = 2$, $f_2 = 3$: $\mathrm{S} \xrightarrow{1} \mathrm{E}$, $\mathrm{S} \xrightarrow{2} \mathrm{G}$, $\mathrm{E} \xrightarrow{1} \mathrm{G}$
For $n = 3$, $f_3 = 5$: $\mathrm{S} \xrightarrow{1} \mathrm{E}$, $\mathrm{S} \xrightarrow{2} \mathrm{F}$, $\mathrm{S} \xrightarrow{1} \mathrm{G}$, $\mathrm{F} \xrightarrow{2} \mathrm{G}$, $\mathrm{E} \xrightarrow{1} \mathrm{G}$

(c) Let $s(p,q)$ be the number of moves for G(p, q, S, E, F, G). The recursive step in the problem is described for $p > 0$, so the simplest case is $p = 0$ and $s(0,q) = h(q) = 2^q - 1$. In that case, (i) tells us what to do.

Otherwise, the number of moves in (ii) is $s(p,q) = 2s(i,j) + h_q$. To find the minimum, we look at all allowed values of i and j, choose those for which $s(i,j)$ is a minimum. This choice of i and j, when used in (ii) tells us which moves to make. In the following table, numbers on the rows refer to p and those on the columns refer to q. Except for the s_p column, then entries are $s(p,q)$. The $p = 0$ row is h_q by (i). To find $s(p,q)$ for $p > 0$, we use (ii). To do this, we look along the diagonal whose indices sum to p, choose the minimum (It's location is (i,j).), double it and add h_q. For example, $s(5,2)$ is found by taking the minimum of the diagonal entries at (0,5), (1,4), (2,3), (3,2), and (4,1). Since these entries are 31, 17, 13, 13, and 19, the minimum is 13. Since this occurs at (2,3) and (3,2), we have a choice for (i,j). Either one gives us $2 \times 13 + h_2 = 29$ moves. To compute s_n we simply look along the $p + q = n$ diagonal and choose the minimum.

	s_p	1	2	3	4	5	6	(values of q)
0		1	3	7	15	31	63	$(s(0,q) = h_q)$
1	1	3	5	9	17	33	65	
2	3	7	9	13	21	27		
3	5	11	13	17	25			
4	9	19	21	25				
5	13	27	29					
6	17	35						Column labels are p.

(d) From the description of the algorithm,

- $s(p,q) = 2 \min s(i,j) + h_q$, where the minimimum is over $i + j = p$ and
- $s_n = \min s(p,q)$, where the minimum is over $p + q = n$.

Putting these together gives us $s(p,q) = 2s_p + h_q$ and so $s_n = \min(2s_p + h_q)$. The initial condition is $s_0 = 0$. In summary

$$s_n = \begin{cases} 0 & \text{if } n = 0, \\ \min\limits_{\substack{p+q=n \\ q>0}} (2s_p + h_q) & \text{if } n > 0. \end{cases}$$

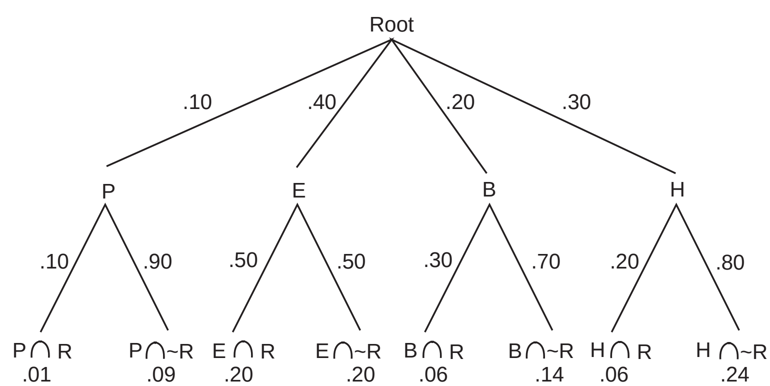
(e) Change the recursive procedure in the algorithm to use the moves for f_p instead of using those for $s(p,q)$. It follows that we can solve the puzzle in $2f_{n-j} + h_j$ moves.

DT-4.1 When there is replacement, the result of the first choice does not matter since the ball is placed back in the box. Hence the answer to both parts of (a) is 3/7.

(b) If the first ball is green, we are drawing a ball from three white and three green and so the probability is $3/6 = 1/2$. If the first ball is white, we are drawing a ball from two white and four green and so the probability is $2/6 = 1/3$.

DT-4.2 There are five ways to get a total of six: $1+5$, $2+4$, $3+3$, $4+2$, and $5+1$. All five are equally likely and so each outcome has probability 1/5. We get the answers by counting the number that satisfy the given conditions and multiplying by 1/5:
(a) 1/5, (b) 2/5, (c) 3/5.

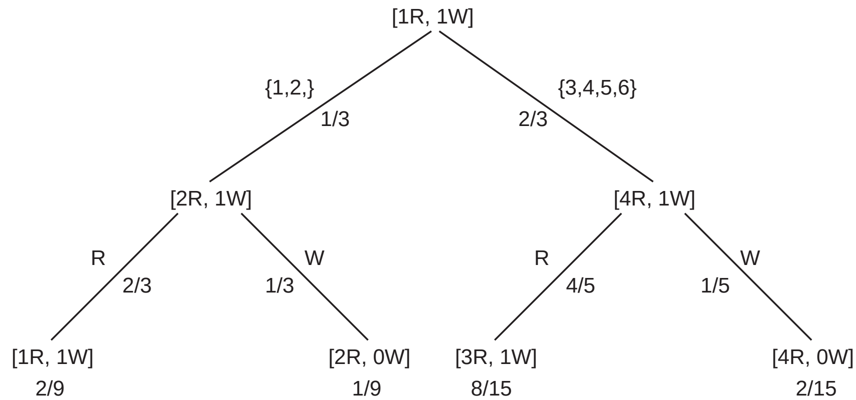
DT-4.3 Here is the decision tree for this problem



(a) We want to compute the conditional probability that a student is a humanities major, given that that student has read Hamlet. In the decision tree, if we follow the path from the Root to H to $H \cap R$, we get a probability of .06 at the leaf. We must divide this by the sum over all probabilities of such paths that end at $X \cap R$ (as opposed to $X \cap \sim R$). That sum is $0.01 + 0.20 + 0.06 + 0.06 = 0.33$. The answer is $0.06/0.33 = 0.182$.

(b) We compute the probabilities that a student has not read Hamlet and is a P (Physical Science) or E (Engineering) major: $0.09 + 0.20 = 0.29$. We must divide this by the sum over all probabilities of such paths that end at $X \cap \sim R$ (as opposed to $X \cap R$). The answer is $0.29/0.67 = 0.433$.

DT-4.4 Here is a decision tree where the vertices are urn compositions. The edges incident on the root are labeled with the outcome sets of the die and the probabilities that these sets occur. The edges incident on the leaves are labeled with the color of the ball drawn and the probability that such a ball is drawn. The leaves are labeled with the product of the probabilities on the edges leading from the root to that leaf.

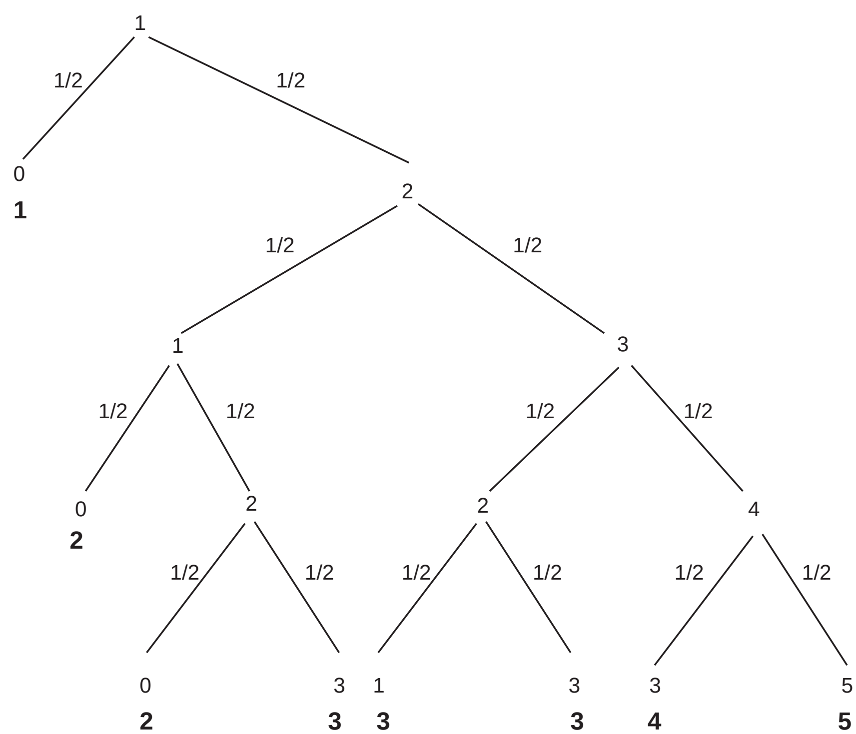


(a) To compute the conditional probability that a 1 or 2 appeared, given that a red ball was drawn, we take the probability 2/9 that a 1 or 2 appeared and a red ball was drawn and divide by the total probability that a red ball was drawn: $2/9 + 8/15 = 34/45$. The answer is $5/17 = 0.294$.

(b) We divide the probability that a 1 or 2 appeared and the final composition had more than one red ball (1/9) by the sum of the probabilities where the final composition had more than one red ball : $1/9 + 8/15 + 2/15 = 7/9 = 0.78$.

DT-4.5 A decision tree is shown below. The values of the random variable X are shown just below the amount remaining in the pot associated with each leaf. To compute $E(X)$ we sum the values of X times the product of the probabilities along the path from the root to that value of X. Thus, we get

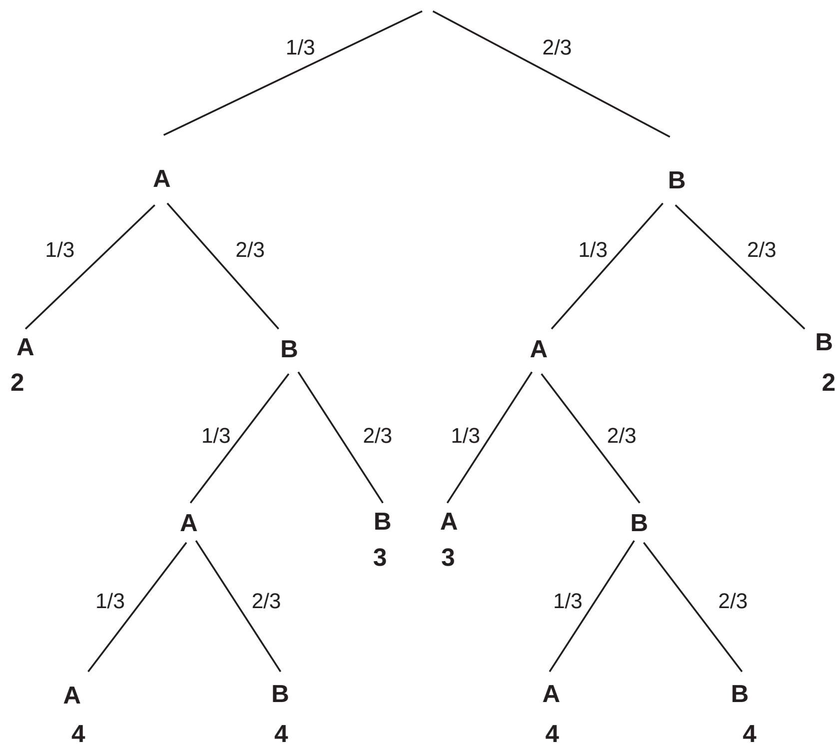
$$E(X) = 1 \times (1/2) + 2 \times (1/8) + (2 + 3 + 3 + 3 + 4 + 5) \times (1/16) = 2\,.$$



DT-4.6 A decision tree is shown below. Under the leaves is the length of the game (the height of the leaf). The expected length of the game is the sum of the products of the probabilities on the edges of each path to a leaf times the height of that leaf:

$$2((1/3)^2 + (2/3)^2)+$$
$$4((1/3)^3(2/3) + (1/3)^2(2/3)^2 + (1/3)^2(2/3)^2 + (1/3)(2/3)^3)+$$
$$3((1/3)(2/3)^2 + (1/3)^2(2/3)\,.$$

The expected number of games is about 2.69.



DT-4.7 Let p_k denote the probability that the gambler is ruined if he starts with $0 \le k \le Q$ dollars. Note that $p_0 = 1$ and $p_Q = 0$. Assume $1 < k \le Q$. Then the recurrence relation $p_{k-1} = (1/2)p_k + (1/2)p_{k-2}$ holds. Solving for p_k gives $p_k = 2p_{k-1} - p_{k-2}$. This looks familiar. It is a two term linear recurrence relation. But the setup was a little strange! We would expect to know p_0 and p_1 and would expect the values of p_k to make sense for all $k \ge 0$. But here we have an interpretation of the p_k only for $0 \le k \le Q$ and we know p_0 and p_Q instead of p_0 and p_1. Such a situation is not for faint-hearted students.

We are going to keep going as if we knew what we were doing. The characteristic equation is $r^2 - 2r + 1 = 0$. There is one root, $r = 1$. That means that the sequence $a_k = 1$, for all $k = 0, 1, 2, \ldots$, is a solution and so is $b_k = k$, for $k = 0, 1, 2, \ldots$. We need to find A and B such that $Aa_0 + Bb_0 = 1$ and $Aa_Q + Bb_Q = 0$. We find that $A = 1$ and $B = -1/Q$. Thus we have the general solution

$$p_k = 1 - \frac{k}{Q} = \frac{Q-k}{Q} \qquad q_k = \frac{k}{Q}\,.$$

Note that p_k is defined for all $k \geq 0$ like it would be for any such linear two term recurrence. The fact that we are only interested in it for $0 \leq k \leq Q$ is no problem to the theory.

Suppose a rich student, Brently Q. Snodgrass the III, has $8,000$ dollars and he wants to play the coin toss game to make $10,000$ dollars so he has $2,000$ his parents don't know about. His probabililty of being ruined is $(10,000 - 8000)/10000 = 1/5$. His probability of getting his extra 2000 dollars is $4/5$. A poor student who only had 100 dollars and wanted to make 2000 dollars would have a probability of $(2,100 - 100)/2,100 = 0.95$ of being ruined. Life isn't fair.

There is one consolation. The expected number of times Brently will have to toss the coin to earn his 2,000 dollars is $16,000,000$. It will take him 69.4 weeks tossing 40 hours per week, one toss every 10 seconds. If he does get his 2000 dollars, he will have been working as a "coin tosser" for over a year at a salary of 72 cents per hour. He should get a minimum wage job instead!