Solutions for Basic Concepts in Graph Theory

GT-1.1 To specify a graph we must choose $E \in \mathcal{P}_2(V)$. Let $N = |\mathcal{P}_2(V)|$. (Note that $N = \binom{n}{2}$.) There are 2^N subsets E of $\mathcal{P}_2(V)$ and $\binom{N}{q}$ of them have cardinality q. This proves (a) and answers (b).

GT-1.2 The sum is the number of ends of edges since, if x and y are the ends of an edge, the edge contributes 1 to the value of $d(x)$ and 1 to the value of $d(y)$. Since each edge has two ends, the sum is twice the number of edges.

Since $\sum_v d(v)$ is even if and only if the number of odd summands is even, it follows that there are an even number of v for which $d(v)$ is odd.

GT-1.3 (a) The graph is isomorphic to Q. The correspondence between vertices is given by

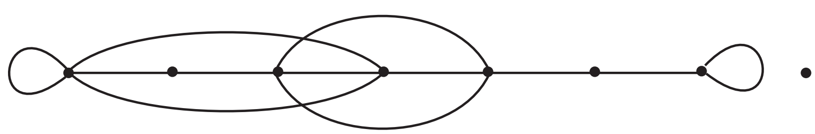
$$\phi = \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & A & C & E & F & D & G & B \end{pmatrix}$$

where the top row corresponds to the vertices of Q.

(b) The graph Q' is not ismorphic to Q. It can be made isomorphic by deleting one edge and adding another. You should try to figure out which edges these are.

GT-1.4 (a) $(0, 2, 2, 3, 4, 4, 4, 5)$ is the degree sequence of Q. (b) If a pictorial representation of R can be created by labeling $P'(Q)$ with the edges and vertices of R, then R has degree sequence $(0, 2, 2, 3, 4, 4, 4, 5)$ because the degree sequence is determined by ϕ.

(c) This is the converse of (b). It is false. The following graph has degree sequence $(0, 2, 2, 3, 4, 4, 4, 5)$ but cannot be morphed into the form $P'(Q)$.



GT-1.5 (a) There is no graph Q with degree sequence $(1, 1, 2, 3, 3, 5)$ since the sum of the degrees is odd. The sum of the degrees of a graph is $2|E|$ and must, therefore, be even.

(d) (answers (b) and (c) as well) There is a graph with degree sequence $(1, 2, 2, 3, 3, 5)$, no loops or parallel edges allowed. Take

$$\phi = \begin{pmatrix} a & b & c & d & e & f & g & h \\ A & B & C & A & B & C & E & F \\ B & C & E & D & D & D & D & D \end{pmatrix}.$$

(e) (answers (f) as well) A graph with degree sequence $(3, 3, 3, 3)$ has $(3+3+3+3)/2 = 6$ edges and, of course 4 vertices. That is the maximum $\binom{4}{2}$ of edges that a simple graph with 4 vertices can have. It is easy to construct such a graph. Draw the four vertices and make all possible connections. This graph is called the *complete* graph on 4 vertices.

(g) There is no simple graph (or graph without loops or parallel edges) with degree sequence $(3,3,3,5)$. See (f).

(h) Similar arguments to (f) apply to the complete graph with degree sequence $(4,4,4,4,4)$. Such a graph would have $20/2 = 10$ edges. But $\binom{5}{2} = 10$. To construct such a graph, use 5 vertices and make all possible connections.

(i) There is no such graph. See (h).

GT-1.6 Each of (a) and (c) has just one pair of parallel edges (edges with the same endpoints), while (b) and (d) each have two pairs of parallel edges. Thus neither (b) nor (d) is equivalent to (a) or (c). Vertex 1 of (b) has degree 4, but (d) has no vertices of degree 4. Thus (b) and (d) are not equivalent. It turns out that (a) and (c) are equivalent. Can you see how to make the forms correspond?

GT-1.7 (a) We know that the expected number of triangles behaves like $(np)^3/6$. This equals 1 when $p = 6^{1/3}/n$.

(b) By Example 5, the expected number of edges is $\binom{n}{2}p$, which behaves like $(n^2/2)p$ for large n. Thus we expect about $(6^{1/3}/2)n$

GT-1.8 Introduce random variables X_S, one for each $S \in \mathcal{P}_k(V)$. Reasoning as in the example, $E(X_S) = p^K$ where $K = \binom{k}{2}$, the number of edges that must be present. Thus the expected number of sets of k vertices with all edges present is $\binom{n}{k}p^K$.

For large n, this behaves like $n^k p^K/k!$, which will be 1 when $p = (k!/n^k)^{1/K}$. For large n, the exected number of edges behaves like $(n^2/2)(k!/n^k)^{1/K}$. This last number has the form Cn^α where $C = (k!)^{1/K}/2$ and $\alpha = 2 - k/K = 2 - 2/(k-1) = \frac{2(k-2)}{k-1}$.

GT-1.9 The first part comes from factoring out $\binom{n}{3}p^3$ from the last equation in Example 6. To obtain the inequality, replace $(1-p^3)$ with $(1-p^2)$, factor it out, and use $1+3(n-3) < 3n$.

GT-2.1 Since $E \subseteq \mathcal{P}_2(V)$, we have a simple graph. Regardless of whether you are in set C or S, following an edge takes you into the other set. Thus, following a path with an odd number of edges takes you to the opposite set from where you started while a path with an even number of edges takes you back to your starting set. Since a cycle returns to its starting vertex, it obviously returns to its starting set.

GT-2.2 (a) The graph is not Eulerian. The longest trail has 5 edges, the longest circuit has 4 edges.

(b) The longest trail has 9 edges, the longest circuit has 8 edges.

(c) The longest trail has 13 edges (an Eulerian trail starting at C and ending at D). The longest circuit has 12 edges (remove edge f).

(d) This graph has an Eulerian circuit (12 edges).

GT-2.3 (a) The graph is Hamiltonian.

(b) The graph is Hamiltonian.

(c) The graph is not Hamiltonian. There is a cycle that includes all vertices except K.

(d) The graph is Hamiltonian.

GT-2.4 (a) There are $|V \times V|$ potential edges to choose from. Since there are two choices for each edge (either in the digraph or not), we get 2^{n^2} simple digraphs.

(b) With loops forbidden, our possible edges include all elements of $V \times V$ except those of the form (v, v) with $v \in V$. Thus there are $2^{n(n-1)}$ loopless simple digraphs. An alternative derivation is to note that a simple graph has $\binom{n}{2}$ edges and we have 4 possible choices in constructing a digraph: (i) omit the edge, (ii) include the edge directed one way, (iii) include the edge directed the other way, and (iv) include two edges, one directed each way. This gives $4^{\binom{n}{2}} = 2^{n(n-1)}$. The latter approach is not useful in doing part (c).

(c) Given the set S of possible edges, we want to choose q of them. This can be done in $\binom{|S|}{q}$ ways. In the general case, the number is $\binom{n^2}{q}$ and in the loopless case it is $\binom{n(n-1)}{q}$.

GT-2.5 (a) Let $V = \{u, v\}$ and $E = \{(u, v), (v, u)\}$.

(b) For each $\{u, v\} \in \mathcal{P}_2(V)$ we have three choices: (i) select the edge (u, v), (ii) select the edge (v, u) or (iii) have no edge between u and v. Let $N = |\mathcal{P}_2(V)| = \binom{n}{2}$. There are 3^N oriented simple graphs.

(c) We can choose q elements of $\mathcal{P}_2(V)$ and then orient each of them in one of two ways. This gives us $\binom{N}{q}2^q$.

GT-2.6 (a) For all $x \in S$, $x|x$. For all $x, y \in S$, if $x|y$ and $x \neq y$, then y does not divide x. For all $x, y, z \in S$, $x|y$, $y|z$ implies that $x|z$.

(b) The covering relation is

$$\begin{aligned} H = \{&(2,4),(2,6),(2,10),(2,14),(3,6),(3,9),(3,15),\\ &(4,8),(4,12),(5,10),(5,15),(6,12),(7,14)\}. \end{aligned}$$

We leave it to you to draw the picture!

GT-3.1 (a) Suppose G is a connected graph with v vertices and v edges. A connected graph is a tree if and only if the number of vertices is one more than the number of edges. Thus G is not a tree and must have at least one cycle. This proves the base case, $n = 0$. Suppose $n > 0$ and G is a graph with v vertices and $v + n$ edges. We know that the graph is not a tree and thus has a cycle. We know that removing an edge from a cycle does not disconnect the graph. However, removing the edge destroys any cycles that contain it. Hence the new graph G' contains one less edge and at least one less cycle than G. By the induction hypothesis, G' has at least n cycles. Thus G has at least $n + 1$ cycles.

(b) Let G be a graph with components $G_1, \ldots, G_k$. With subscripts denoting components, G_i has v_i vertices, $e_i = v_i + n_i$ edges and at least $n_i + 1$ cycles. From the last two formulas, G_i has at least $1 + e_i - v_i$ cycles. Now sum over i.

(c) For each n we wish to construct a simple graph that has n more edges than vertices but has only $n + 1$ cycles. There are many possibilities. Here's one solution. The vertices are v and, for $0 \le i \le n$, x_i and y_i. The edges are $\{v, x_i\}$, $\{v, y_i\}$, and $\{x_i, y_i\}$. (This gives $n+1$ triangles joined at v.) There are $1+2(n+1)$ vertices, $3(n+1)$ edges, and $n + 1$ cycles.

GT-3.2 (a) $\sum_{v \in V} d(v) = 2|E|$. For a tree, $|E| = |V| - 1$. Since $2|V| = \sum_{v \in V} 2$,

$$2 = 2|V| - 2|E| = \sum_{v \in V} (2 - d(v)).$$

(b) Suppose that T is more than just a single vertex. Since T is connected, $d(v) \neq 0$ for all v. Let n_k be the number of vertices of T of degree k. By the previous result, $\sum_{k\geq 1}(2-k)n_k = 2$. Rearranging gives $n_1 = 2 + \sum_{k\geq 2}(k-2)n_k$. If $n_m \geq 1$, the sum is at least $m-2$.

(c) Let the vertices be u and v_i for $1 \leq i \leq m$. Let the edges be $\{u, v_i\}$ for $1 \leq i \leq m$.

GT-3.3 (a) No such tree exists. A tree with six vertices must have five edges.

(b) No such tree exists. Such a tree must have at least one vertex of degree three or more and hence at least three vertices of degree one.

(c) A graph with two connected components, each a tree, each with five vertices will have this property.

(d) No such graph exists.

(e) No such tree exists.

(f) Such a graph must have at least $c + e - v = 1 + 6 - 4 = 3$ cycles.

(g) No such graph exists. If the graph has no cycles, then each component is a tree. In such a graph, the number of vertices is strictly greater than the number of edges for each component and hence for the whole graph.

GT-3.4 (a) The idea is that for a rooted planar tree of height h, having at most 2 children for each non-leaf, the tree with the most leaves occurs when each non-leaf vertex has exactly 2 children. You should sketch some cases and make sure you understand this point. For this case $l = 2^h$ and so $\log_2(l) = h$. Any other rooted planar tree of height h, having most 2 children for each non-leaf, is a subtree (with the same root) of this maximal-leaf binary tree and thus has fewer leaves.

(b) Knowing the number of leaves does not bound the height of a tree — it can be arbitrarily large.

(c) The maximum height is $h = l - 1$. One leaf has height 1, one height 2, etc., one of height $l-2$ and, finally, two of height $l-1$.

(d) (answers (e) as well) $\lceil \log_2(l) \rceil$ is a lower bound for the height of *any* binary tree with l leaves. It is easy to see that you can construct a full binary tree with l leaves and height $\lceil \log_2(l) \rceil$.

GT-3.5 (a) A binary tree with 35 leaves and height 100 is possible.

(b) A full binary tree with 21 leaves can have height at most 20. So such a tree of height 21 is impossible.

(c) A binary tree of height 5 can have at most 32 leaves. So one with 33 leaves is impossible.

(d) No way! The total number of vertices is

$$\sum_{i=0}^{5} 3^5 = \frac{3^6 - 1}{2} = 364\,.$$

GT-3.6 (a) For *(1)* there are four spanning trees. For *(2)* there are 8 spanning trees. Note that there are $\binom{5}{3} = 10$ ways to choose three edges. Eight of these 10 choices result in spanning trees, the other two choices result in cycles (with vertex sequences (A, B, D) and (B, C, D)). For *(3)* there are 16 spanning trees.

(b) For *(1)* there is one. For *(2)* there are two. For *(3)* there are two.

(c) For *(1)* there are two. For *(2)* there are four. For *(3)* there are six.

(d) For *(1)* there are two. For *(2)* there are three. For *(3)* there are six.

GT-3.7 (a) For *(1)* there are three minimal spanning trees. For *(2)* there are two minimal spanning trees. For *(3)* there is one minimal spanning tree.

(b) For *(1)* there is one minimal spanning tree up to isomorphism. For *(2)* there are two. For *(3)* there is one.

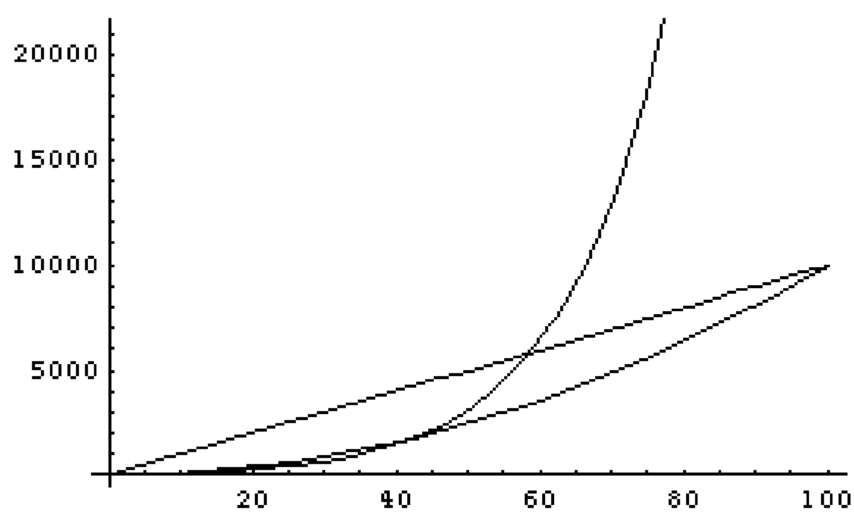
(c) For *(1)* there is one. For *(2)* there is one. For *(3)* there are four.

(d) For *(1)* there are two. For *(2)* there is one. For *(3)* there are four.

GT-3.8 (a) (and (b)) There are 21 vertices, so the minimal spanning tree has 20 edges. Its weight is 30. We omit details.

(c) Note that K is a the only vertex in common to the two bicomponents of this graph. Whenever this happens (two bicomponents, common vertex), the depth-first spanning tree rooted at that common vertex has exactly two "principal subtrees" at the root. In other words, the root of the depth-first spanning tree has down-degree two (two children). The two children of K can be taken to be P and L. P is the root of a subtree consisting of 5 vertices, 4 with one child, one leaf. L is the root of a subtree consisting of 15 vertices, 14 with one child, one leaf.

GT-4.1 (a) The algorithm that has running time $100n$ is better than the one with running time n^2 for $n > 100$. $100n$ is better than $(2^{n/10} - 1)100$ for $n \geq 60$. For $1 \leq n < 10$, $(2^{n/10} - 1)100$ is worse than n^2. At $n = 10$ they are the same. For $10 < n < 43$, n^2 is worse than $(2^{n/10} - 1)100$. For $n \geq 43$, $(2^{n/10} - 1)100$ is worse than n^2. Here are the graphs:



(b) When n is very large, B is fastest and C is slowest. This is because, of two polynomials the one with the lower degree is eventually faster and an exponential function grows faster than any polynomial.

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GT-4.2 (a) The most direct way to prove this is is to use Example 21. additional observations on Θ and O.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = C > 0 \quad \text{implies} \quad g(n) \quad \text{is} \quad \Theta(f(n))$$

Let $p(n) = \sum_{i=0}^{k} b_i n^i$ with $b_k > 0$. Take $f(n) = p(n)$, $g(n) = n^k$ and $C = b_k > 0$. Thus, $p(n)$ is $\Theta(n^k)$, hence the equivalence class of each is the same set: $\Theta(p(n))$ is $\Theta(n^k)$.

(b) $\mathrm{O}(p(n))$ is $\mathrm{O}(n^k)$ follows from (a).

(c) $\lim_{n\to\infty} p(n)/a^n = 0$. This requires some calculus. By applying l'Hospital's Rule k times, we see that the limit is $\lim_{n\to\infty} (k!/(\log(a))^k)/a^n$, which is 0. Any algorithm with exponential running time is eventually much slower than a polynomial time algorithm.

(d) For $p(n)$ to be $\Theta(a^{Cn^k})$, we must have positive constants A and B such that $A \le a^{p(n)}/a^{Cn^k} \le B$. Taking logarithms gives us $\log_a A \le p(n) - Cn^k \le \log_a B$. The center of this expression is a polynomial which is not constant unless $p(n) = Cn^k + D$ for some constant D, the case which is ruled out. Thus $p(n) - Cn^k$ is a nonconstant polynomial and so is unbounded.

GT-4.3 Here is a general method of working this type of problem:

Let $p(n) = \sum_{i=0}^{k} b_i n^i$ with $b_k > 0$. Show using definition that $\Theta(p(n))$ is $\Theta(n^k)$. Let $s = \sum_{i=0}^{k-1} |b_i|$ and assume that $n \ge 2s/b_k$. We have

$$|p(n) - b_k n^k| \le \left| \sum_{i=0}^{k-1} b_i n^i \right| \le \sum_{i=0}^{k-1} |b_i| n^i \le \sum_{i=0}^{k-1} |b_i| n^{k-1} = s n^{k-1} \le b_k n^k/2.$$

Thus $|p(n)| \ge b_k n^k - b_k n^k/2 \ge (b_k/2)n^k$ and also $|p(n)| \le b_k n^k + b_k n^k/2 \le (3b_k/2)n^k$.

The definition is satisfied with $N = 2s/b_k$, $A = (b_k/2)$ and $B = (3b_k/2)$. If you want to show, using the definition, that $\Theta(p(n))$ is $\Theta(Kn^k)$ for some $K > 0$, replace A with $A' = A/K$ and B with $B' = B/K$.

In our particular cases we can be sloppy and it gets easier. Take (a) as an example.

(a) For $g(n) = n^3 + 5n^2 + 10$, choose N such that $n^3 > 5n^2 + 10$ for $n > N$. You can be ridiculous in the choice of N. $N^3 > 5N^2 + 10$ is valid if $1 > 5/N + 10/N^3$. $N = 10$ is plenty big enough. If $n^3 > 5n^2 + 10$ then $n^3 < g(n) < 2n^3$. So taking $A = 1$ and $B = 2$ works for the definition: $An^3 < g(n) < Bn^3$ showing g is $\Theta(n^3)$. If you want to use $f(n) = 20n^3$ as the problem calls for, replace these constants by $A' = A/20$ and $B' = B/20$. Thus, $A'(20n^3) < g(n) < B'(20n^3)$ for $n > N$.

This problem should make you appreciate the much easier approach of Example 21.

GT-4.4 (a) There is an explicit formula for the sum of the squares of integers.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

This is a polynomial of degree 3, hence the sum is $\Theta(n^3)$.

(b) There is an explicit formula for the sum of the cubes of integers.

$$\sum_{i=1}^{n} i^3 = \Big(\frac{n(n+1))}{2}\Big)^2 .$$

This is a polynomial of degree 4, hence the sum is $\Theta(n^4)$.

(c) To show the $\sum_{i=1}^{n} i^{1/2}$ is $\Theta(n^{3/2})$ it helps to know a little calculus. You can interpret the integral as upper and lower Riemann sum approximations to the integral of $f(x) = x^{1/2}$ with $\Delta x = 1$:

$$\int_0^n f(x)\,dx < \sum_{i=1}^{n} i^{1/2} = \sum_{i=1}^{n-1} i^{1/2} + n^{1/2} < \int_1^n f(x)\,dx + n^{1/2}.$$

Since $\int x^{1/2}\,dx = 2x^{3/2}/3 + C$. You can fill in the details to get $\Theta(n^{3/2})$.

The method used in (c) will also work for (a) and (b). The idea works in general: Suppose $f(x) \geq 0$ and $f'(x) > 0$. Let $F(x)$ be the antiderivative of $f(x)$. If $f(n)$ is $\mathrm{O}(F(n))$, then $\sum_{i=0}^{n} f(n)$ is $\Theta(F(n))$. There is a similar result if $f'(x) < 0$: replace "$f(n)$ is $\mathrm{O}(F(n))$" with "$f(1)$ is $\mathrm{O}(F(n))$."

GT-4.5 (a) To show $\sum_{i=1}^{n} i^{-1}$ is $\Theta(\log_b(n))$ for any base $b > 1$ use the Riemann sum trick from the previous exercise. $\int_1^n x^{-1}\,dx = \ln(x)$. This shows that $\sum_{i=1}^{n} i^{-1}$ is $\Theta(\log_e(n))$. But, $\log_e(x) = \log_e(b)\log_b(x)$ (as we learned in high school). Thus, $\log_e(x)$ and $\log_b(x)$ belong to the same Θ equivalence class as they differ by a postive constant multiple $\log_e(b)$ (recall $b > 1$).

(b) First you need to note that $\log_b(n!) = \sum_{i=1}^{n} \log_b(i)$. Use the Riemann sum trick again.

$$\int_1^n \log_b(x)\,dx = \log_b(e)\int_1^n \log_e(x)\,dx = \log_b(e)\big(n\ln(n) - n + 1\big)\,.$$

Thus, the sum is $\Theta(n\ln(n) - n + 1)$ which is $\Theta(n\ln(n))$ which is $\Theta(n\log_b(n))$.

(c) Use Stirling's approximation for $n!$, $n!$ is asymptotic to $(n/e)^n(2\pi n)^{1/2}$. Thus, $n!$ is $\Theta((n/e)^n(2\pi n)^{1/2})$, by Example 21. Do a little algebra to rearrange the latter expression to get $\Theta((n/e)^{n+1/2})$.