Trigonometric Integrals

Suppose you have an integral that just involves trig functions. It is usually possible to use trig identities to get it so all the trig functions have the same argument, say x. Here are a couple of ways to change that integral.

The half-angle substitution Let $u = \tan(x/2)$. Then a bit of trigonometry and a little calculus gives

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2} \quad \text{and} \quad dx = \frac{2\,du}{1+u^2}.$$

If the original integral was a rational function of trig functions, the substitution gives a rational function that can be integrated using partial fractions.

The complex exponential Use the identities in the complex numbers supplement to get everything in terms of e^{ix}. Now set $u = e^{ix}$. Thus

$$\sin x = \frac{u - 1/u}{2i}, \quad \cos x = \frac{u + 1/u}{2} \quad \text{and} \quad dx = \frac{-i\,du}{u}.$$

If the original integral was a rational function of trig functions, the substitution gives a rational function that can be integrated using partial fractions.

Example The derivation of the integral of the secant is usually done by the trick of introducing the factor $\sec x = \tan x$. Let's use the half-angle substitution.

$$\int \sec x\, dx = \int \frac{1+u^2}{1-u^2}\frac{2\,du}{1+u^2} = \int \frac{2\,du}{1-u^2} = \int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du = \ln\left|\frac{1+u}{1-u}\right| + C$$

Substituting back and using trig identities gives the answer. First multiplying by $\frac{1+u}{1+u}$ makes the calculations a bit simpler.

Example Suppose we want $\displaystyle\int \frac{\cos x}{2+\cos x} dx$. By the first method, this is

$$\int \frac{\frac{1-u^2}{1+u^2}}{2+\frac{1-u^2}{1+u^2}}\frac{2\,du}{1+u^2} \;=\; 2\int \frac{1-u^2}{(1+u^2)(3+u^2)}du,$$

which we won't go any further with. By the second method, this is

$$\int \frac{\frac{u+1/u}{2}}{2+\frac{u+1/u}{2}}\frac{-i\,du}{u} \;=\; -i\int \frac{u^2+1}{u(u^2+4u+1)}du,$$

which we won't go any further with.