We want to prove that

$$\int_C \mathbf{G}\cdot d\mathbf{R} \;=\; \int\!\!\int_S (\nabla\times\mathbf{G})\cdot d\mathbf{S}.$$

Recall that x, y, z are parameterized by u, v on the surface and u, v are parameterized by t on the boundary. To get the surface S, the parameters u and v must take on all the values in some domain Σ in the uv-plane. Let Γ be the boundary of this region. Since

$$\frac{d\mathbf{R}}{dt} \;=\; \frac{\partial\mathbf{R}}{\partial u}\frac{du}{dt} + \frac{\partial\mathbf{R}}{\partial v}\frac{dv}{dt},$$

we have

$$\int_C \mathbf{G}\cdot d\mathbf{R} \;=\; \int_\Gamma \left(\mathbf{G}\cdot\frac{\partial\mathbf{R}}{\partial u}\frac{du}{dt} + \mathbf{G}\cdot\frac{\partial\mathbf{R}}{\partial v}\frac{dv}{dt}\right) dt \;=\; \int_\Gamma \left(\mathbf{G}\cdot\frac{\partial\mathbf{R}}{\partial u}\right) du + \left(\mathbf{G}\cdot\frac{\partial\mathbf{R}}{\partial v}\right) dv.$$

By Green's Theorem in the uv-plane with the first parenthesized term being F_1 and the second being F_2, this integral is

$$\int\!\!\int_\Sigma \left\{\frac{\partial}{\partial u}\left(\mathbf{G}\cdot\frac{\partial\mathbf{R}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{G}\cdot\frac{\partial\mathbf{R}}{\partial u}\right)\right\} du\,dv.$$

Expanding the partial of products, we finally get

$$\frac{d\mathbf{R}}{dt} \;=\; \int\!\!\int_\Sigma \left(\frac{\partial\mathbf{G}}{\partial u}\cdot\frac{\partial\mathbf{R}}{\partial v} - \frac{\partial\mathbf{G}}{\partial v}\cdot\frac{\partial\mathbf{R}}{\partial u}\right) du\,dv.$$

Since $d\mathbf{S} = (\partial\mathbf{R}/\partial u)\times(\partial\mathbf{R}/\partial v)du\,dv$, we would like to prove that

$$\frac{\partial\mathbf{G}}{\partial u}\cdot\frac{\partial\mathbf{R}}{\partial v} - \frac{\partial\mathbf{G}}{\partial v}\cdot\frac{\partial\mathbf{R}}{\partial u} \;=\; (\nabla\times\mathbf{G})\cdot\left(\frac{\partial\mathbf{R}}{\partial u}\times\frac{\partial\mathbf{R}}{\partial v}\right).$$

This is where all the vector identity stuff gets used.