A vector (or scalar) function $\mathbf{F}(\mathbf{R})$ is called *homogeneous of degree* d if $\mathbf{F}(t\mathbf{R}) = t^d\mathbf{F}(\mathbf{R})$ for all vectors $\mathbf{R}$ and real numbers t such that $\mathbf{R}$ and $t\mathbf{R}$ are in the domain of $\mathbf{F}$. (If $d = 0$, we define $0^d = 1$.)

S1. Suppose $\mathbf{F}(\mathbf{R})$ is defined in a domain that is star shaped with respect to the origin and is homogenous of degree d. Using the integral formulas from Sections 4.4 and 4.5 with $\mathbf{R}_0 = \mathbf{0}$, prove the following:

(a) If $\mathbf{F}$ has a scalar potential, then $\frac{1}{d+1}\mathbf{F}(\mathbf{R}) \cdot \mathbf{R}$ is a scalar potential for $\mathbf{F}$ and it is homogeneous of degree $d + 1$.

(b) If $\mathbf{F}$ has a vector potential, then $\frac{1}{d+2}\mathbf{F}(\mathbf{R}) \times \mathbf{R}$ is a vector potential for $\mathbf{F}$ and it is homogeneous of degree $d + 1$.

Remark: Since the star-shaped domain contains the origin, we may take $t = 0$ and $\mathbf{R}$ any point in the domain to get $\mathbf{F}(0\mathbf{R}) = 0^d\mathbf{F}(\mathbf{R})$. It follows that $d \geq 0$.

S2. Show that the vector functions $\mathbf{F}$ of Examples 4.9, 4.10 and 4.12 and the vector function in Exercise 4.5.2 are all homogeneous and compute their degrees.

S3. Using Exercises S1 and S2, derive vector (and, for Example 4.12, scalar) potentials for Examples 4.9, 4.10 and 4.12 and Exercise 4.5.2.

S4. For Example 4.12 and Exercise 4.5.2 the vector potentials $\mathbf{G}$ obtained in the previous exercise do not agree with the result in the book. For instance, for Example 4.12 you should obtain $\frac{1}{3}(-yz\mathbf{i} - xz\mathbf{j} + 2xy\mathbf{k})$. Explain why both answers are correct.