# Sherlock Holmes and the Bicycle Tracks

by
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"This track, as you perceive, was made by a rider who was going from the direction of the school."

"Or towards it?"

"No, no, my dear Watson. The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school."

—*The Adventure of the Priory School* by Arthur Conan Doyle

## 1 The Error . . .

Sherlock Holmes's reasoning is flawed: Since the rear wheel follows the front, it will *always* cross over the front wheel unless the cyclist circles around and crosses his own path. Perhaps he'd had some opium recently.

## 2 . . . and the Solution

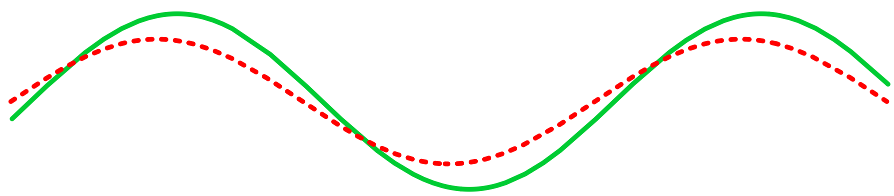
The following observation is the key to determining the direction.

*The rear wheel always moves toward the front wheel.* (1)

**Why is this true?** Because the rear wheel cannot turn. Therefore it moves in the same direction as the bicycle frame is pointed, which is toward the front wheel.

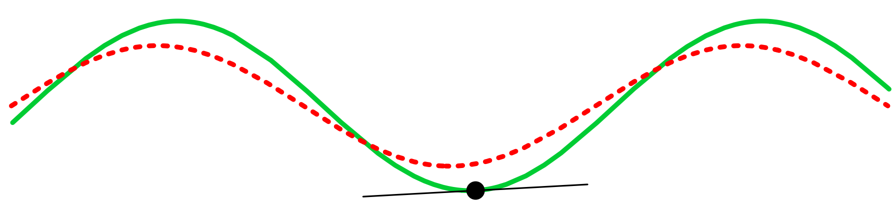
**Why is (1) useful?** At any instant, the direction of motion along a curve is tangent to the curve. Thus we simply draw a tangent to the rear-wheel curve. If we move along it toward where the front wheel was at that time, we will hit the front-wheel curve and the distance between the point on the rear-wheel curve and the point on the front-wheel curve will be the distance between the wheels on the bicycle. If we draw the tangent line to the wrong curve or go in the wrong direction on the tangent line, we may still hit the other curve, but the distance between the two points can be anything and will probably vary as we move along.

The following figure shows the track of a bicycle. We've used different colors for the two tracks and made one of them dashed so that we can easily tell them apart. The picture is roughly ten bicycle lengths long.

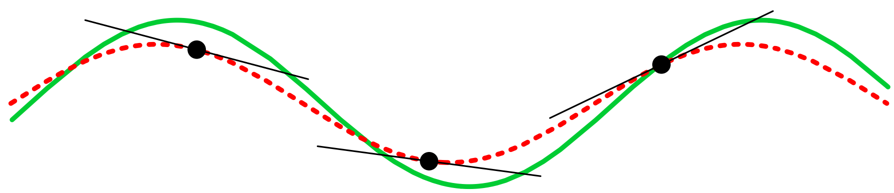


There are at least three ways we could determine that the green track is the front wheel:

- As Holmes noted, the rear wheel track will be deeper, but we can't see that in the picture.
- As we've noted, the rear wheel will cross over the front wheel, which is a bit hard to see in the picture.
- We can use the tangent line idea. If the green track is the rear wheel, a tangent to it will intersect the red track one bicycle length away. Here is a tangent that shows the green track cannot be the rear wheel.



Now we know that the red track is the rear wheel. Which way was the bicycle moving? Here is a picture with some tangents drawn to the red track.



L: If it is moving to the left, then the front wheel will be to the left of the rear wheel. Thus, if we follow a tangent line starting at the red track and moving to the left one bicycle length, we should meet the green track.

R: If the bicycle is moving to the right, then we would move to the right on the tangent line.

It should be easy for you to see that the bicycle was moving to the right.

## 3 How We Drew the Curves

If you like technicalities, you might like to know how we plotted these curves. Let $(x, y) = (f_x(t), f_y(t))$ and $(x, y) = (r_x(t), r_y(t))$ be the positions of the front and rear wheels, respectively, at time $t$. Let $L$ be the distance between the wheels. The slope of the tangent to the rear wheel's path is $dy/dx = r'_y(t)/r'_x(t)$. Because of the way the rear wheel moves,

$$\frac{r'_y(t)}{r'_x(t)} = \frac{f_y(t) - r_y(t)}{f_x(t) - r_x(t)} \quad \text{and} \quad L^2 = (f_y(t) - r_y(t))^2 + (f_x(t) - r_x(t))^2.$$

Because of $r'_x(t)$ and $r'_y(t)$, it is easier to decide on a curve for the rear wheel and then solve these two equations for the position of the front wheel.[1] This involves taking square roots and we need to decide on the sign of the square root. Since we want the bicycle to be moving to the right, $f_x(t) > r_x(t)$,

[1]We might view this as the rear wheel pushes the front wheel along, just as in an actual bicycle. However, this "pushing" is not needed — (1) does not depend on how the bicyle is powered.

which resolves the sign of the square root. After several lines of algebra, we got

$$f_x(t) \;=\; r_x(t) + \frac{L}{\sqrt{1 + (r'_y(t)/r'_x(t))^2}}$$

and

$$f_y(t) \;=\; r_y(t) + \frac{L r'_y(t)/r'_x(t)}{\sqrt{1 + (r'_y(t)/r'_x(t))^2}}.$$

If we are given the coordinates $(r_x(t), r_y(t))$ of the rear wheel as a function of time, we can calculate the position of the front wheels by using these two equations. For our plots we used a sine curve:

$$r_x(t) \;=\; t \quad \text{and} \quad r_y(t) \;=\; A \sin t.$$