1.1 Linear 1st order ODE's

**Def** a 1st order ODE is called "linear" if it can be expressed as

\[ y' + p(x)y = f(x) \]

Why the name?
Shallow reason: it's linear in \( y' \) and \( y \)
Better reason: Think of the equation as

\[ D(y(x)) = f(x) \]

where \( D \) is a function from functions to functions

\[ D\left( g(x) \right) = \frac{d}{dx} g(x) + p(x) g(x) \]

\[ D = \left( \frac{d}{dx} + p(x) \right) \]

\( D \) is called a "differential operator." The point is that \( D \) is a "linear transformation":

\[ D\left( g_1(x) + g_2(x) \right) = D(g_1(x)) + D(g_2(x)) \]

\[ D\left( cg(x) \right) = c D(g(x)) \]

So \( D y(x) = f(x) \) is an ordinary of the matrix-vector equation

\[ A \vec{x} = \vec{b} \]
How to solve it? We run the product rule backwards.

\[ y' + \frac{1}{x} y = \sin(x) \]
\[ xy' + y = x \sin(x) \]
\[ \frac{x}{d} \frac{dy}{dx} + \frac{dy}{dx} = x \sin(x) \]
\[ \frac{d}{dx}(xy) = x \sin(x) \]
\[ xy = \sin(x) - x \cos(x) + C \]
\[ y = \frac{\sin(x)}{x} - \cos(x) + \frac{C}{x} \]

In general:
\[ y' + p(x)y = f(x) \]

Multiply both sides by an "integrating factor" \( r(x) \) so that the left LHS looks like the product rule:
\[ \frac{d}{dx}(r(x)y) = \text{new LHS} = r(x)y' + r(x)p(x)y \]
Note that \[ \frac{d}{dx}(r(x)y) = ry' + r'y \]
So this will work if
\[ r' = rp \]
How to solve it? Product rule in reverse.

**Ex.** \[ y' + y = \sin(x) \]

Let's multiply both sides by a function \( r(x) \), but put off choosing it:

\[ r(x) y' + r(x) y = \sin(x) r(x) \]

LHS looks kind of like \( \frac{d}{dx} (ry) = ry' + r'y \)

and if \( r = e^x \), it really is equal to the LHS:

\[ e^x y' + e^x y = \sin(x) e^x \]

\[ \frac{d}{dx} (e^x y) = \sin(x) e^x \]

\[ e^x y = \int \sin(x) e^x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C \]

\[ y = \frac{1}{2} (\sin x - \cos x) + \frac{C}{e^x} \]
Step 1: Find $r(x)$ with
\[ \frac{d}{dx}(ry) = ry' + rpy \]
(i.e. $r' = rp$)

Step 2: Multiply both sides by $r$
\[ ry' + rpy = rf \]

Step 3: Solve
\[ \frac{d}{dx}(ry) = rf \]
by integrating.

Ex
\[ y' + \frac{y}{x^2} = 1, \quad y(1) = 2 \]

Step 2) need
\[ \frac{d}{dx}(ry) = ry' + r'y = ry' + r' \frac{y}{x^2} \]
\[ r' = \frac{r}{x^2} \]
\[ \frac{dr}{dx} = \frac{r}{x^2} \]
\[ \int \frac{dr}{r} = \int \frac{dx}{x^2} \]
\[ \ln(r) = -\frac{1}{x} + c \]
\[ r = e^{-\frac{1}{x}} \]

Step 2) $y' e^{-\frac{1}{x}} + \frac{y}{x^2} e^{-\frac{1}{x}} = 0 \quad \therefore \quad e^{-\frac{1}{x}}$

Step 3) \[ \frac{d}{dx} \left( y e^{-\frac{x}{2}} \right) = e^{-\frac{x}{2}} \]
\[ ye^{-\frac{1}{x}} = \int e^{-\frac{1}{x}} \, dx \quad y = e^{\int e^{-\frac{1}{x}} \, dx} \]

No way to integrate this, but we can at least incorporate the initial condition.

\[ y = e^{\int_{1}^{x} e^{-\frac{1}{u}} \, du + c} \]

\[ 2 = e^{(0+c)} \]

\[ \frac{2}{e} = c \]

\[ y = e^{\int_{1}^{x} e^{-\frac{1}{u}} \, du + 2e} \]

**Formula for the integrating factor**

\[ \frac{d}{dx} (ry) = r'y + r' \quad ry = ry' + rpy \]

\[ r' = rp(x) \]

\[ \int \frac{dr}{r} = \int p(x) \, dx \]

\[ \ln(r) = \int p(x) \, dx \]

\[ r = e^{\int p(x) \, dx} \]
In fact

So \( \frac{d}{dx} ye = e \int sp(x)da \)

\[ ye = \int e \int sp(x)da \]

\[ y = e \int sp(x)da \]

(1.6) Autonomous Eqn

Def An equation is "autonomous" if it only involves \( y, y', \ldots, y^n \) (no \( x \) is)

E.g. \( \frac{dy}{dx} = y^2, \quad \frac{dx}{dy} = x^2 \)

General 1st order autonomous eqn:

\[ y' = f(y) \]

Suppose \( A \in \mathbb{R}, f(A) = 0 \). Then

\[ y = A \]

solves the solution. This is called an "equilibrium" solution.

Annoyingly, \( A \) is called a "critical point".

Def An equilibrium solution / critical point \( A \) is called "stable" if the solution to

\[ y' = f(y), \quad y(x_0) = A+ \text{(something really small)} \]

satisfies \( \lim_{x \to x_0} y(x) = A \).

Else it is called "unstable"
Ex. \( P'(t) = P(t)(P_{\text{max}} - P(t)) \quad P' = P(P_{\text{max}} - P) \)

Has two equilibrium solutions:

\( P = 0 \quad P = P_{\text{max}} \).

One is stable, one is unstable. Which?

\[ \begin{align*}
\downarrow & \quad P_{\text{max}} \\
\uparrow & \quad 0
\end{align*} \]

6. Could a critical point be stable in one direction and unstable in another?

\[ \begin{align*}
\downarrow & \quad y
\end{align*} \]

7. Yes, if \( f \) is positive (or negative) both above \( A \) and below \( A \)

E.g. \( y' = y^2 \), \( y(0) = k \)

\[ \begin{align*}
\frac{y'}{y^2} &= 1 \\
-\frac{1}{y} &= \alpha + c
\end{align*} \]
\[
-\frac{1}{k} = c
\]
\[
-\frac{1}{y} = \infty - \frac{1}{k}
\]
\[
y = \frac{1}{\frac{1}{k} - \infty} = \frac{k}{1 - k\infty}
\]
(Asymptote at \(\infty = \frac{1}{k}\))

Called "semi-stable" (For us, semi-stable is a special kind of unstable)

**Example:**

\[
y' = e^{y^2} - e\quad y(0) = c. \text{ What is } \lim_{x \to \infty} y(x)?
\]

\[
\int e^{y^2} \, dy = kx. \text{ This is annoying. Instead, let's just find the phase diagram.}
\]
\[
e^{y^2} - 1 = 0
\]

Instead of solving, let's just get the phase diagram.

Critical points:

0 = \(e^{y^2} - e\)

1 = \(y^2\)

\(y = \pm 1\)

\(dy = 0\)

\(e^0 - e < 0\)

at \(y = 0\), \(e^y - e > 0\)

So

\* If \(c < 1\), \(\lim_{x \to \infty} y(x) = \pm 1\)

\* If \(c = 0\), \(\lim_{x \to \infty} y(x) = 1\)

\* If \(c > 1\), \(\lim_{x \to \infty} y(x) = \pm \infty\)