For a second order equation, homogeneous eqn, if \( y_1, y_2 \) are two linearly independent solutions, then every solution can be written as \( c_1 y_1 + c_2 y_2 \).

**Ex.** Find all solutions to \( y'' + y = 0 \)

(i.e. \( y'' = -y \))

\( \cos(x), \sin(x) \) are two solutions, so every solution is of the form \( c_1 \cos(x) + c_2 \sin(x) \).

**Thm.** If \( f, p_1(x) \) and \( p_2(x) \) are continuous, then

\[ y'' + p_1y' + p_2 y = f(x), \quad y(x_0) = b_0, \quad y'(x_0) = b, \]

has a unique solution.

**Ex.** Find all solutions to

\[ y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 2 \]

\[ y = c_1 \sin(x) + c_2 \cos(x) \]

\[ y' = c_1 \cos(x) - c_2 \sin(x) \]

\[ 1 = y(0) = c_1 \]

\[ 2 = y'(0) = c_1 \]

\[ y = 2 \sin(x) + \cos(x) \]
Ex. Find all solutions to
\[ y'' + \frac{1}{x} y' = 1, \quad y(1) = \frac{5}{4} \]

Since there are no \( y \)'s, we can think of this as a first-order equation in \( y' \). It is linear.

Integrating factor is \( e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x \)

\[ axy'' + y' = ax \]
\[ \frac{d}{dx}(axy') = ax \]

\[ axy' = \frac{ax^2}{2} + c_1 \]
\[ y' = \frac{ax}{2} + \frac{c_1}{x} \]
\[ y = \frac{ax^2}{4} + c_1 \ln(x) + c_2 \]

Is this all of the solutions?

- \( \ln(x), 1 \) both solve the equation \( y'' + \frac{1}{x} y' = 0 \)

They are linearly independent.

So yes (here \( \frac{ax^2}{4} \) is the particular solution).

Now, which of these satisfies \( y(1) = \frac{5}{4} \)?

\[ \frac{5}{4} = y(1) = \frac{1}{4} + c_1 \ln(1) + c_2 \]
\[ 1 = c_2 \]

\[ y = \frac{ax^2}{4} + c_1 \ln(x) + 1 \]
"Algorithm" for $y'' + p(x)y' + p_0(x)y = F(x)$, \[\text{[Initial conditions]}\]

1) Find two linearly independent solutions $y_1$ and $y_2$ to $y'' + p_1(x)y' + p_0(x)y = 0$

2) Find a single ("particular") solution $g(x)$ to $y'' + p_1(x)y' + p_0(x)y = F(x)$

3) General solution is $C_1y_1 + C_2y_2 + g$

4) Use initial conditions to solve for $C_1, C_2$

The next section is about step 4.

(2.2) Linear, 2nd order, homogeneous constant coefficients

Let's solve $a_2y'' + a_1y' + a_0y = 0$, $a_2, a_1, a_0 \text{ constants}$.

Guess $e^{\lambda x}$ works for some $\lambda$. We'd need

\[a_2 \frac{d^2}{dx^2} e^{\lambda x} + a_1 \frac{d^2}{dx^2} e^{\lambda x} + a_0 e^{\lambda x} = 0\]

\[a_2 \lambda^2 e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0\]

\[(a_2 \lambda^2 + a_1 \lambda + a_0) e^{\lambda x} = 0\]

Now solve for $\lambda$. 