1. Recall the vector space $V$ from last week’s homework: it is the set of (strictly) positive real numbers with a strange new addition and scalar multiplication defined by

- For any $x, y \in V$, $x \tilde{+} y = xy$
- For any $c \in \mathbb{R}$ and $x \in V$, $c \tilde{\cdot} x = x^c$

(If this doesn’t make sense, take a look back at last week’s written homework). Consider the function $T : \mathbb{R}^1 \to V$ defined by $T([x]) = 2^x$. Your task is to show that $T$ is a linear transformation.

In fact, a little calculus shows that $T$ is one-to-one and onto. Thus, the bizarre vector space $V$ is really not so bizarre - it is just $\mathbb{R}^1$ in disguise.

2. Unlike the last problem, this one is **OPTIONAL**: Understanding the following example would be nice, but I don’t think this sort of thing is remotely essential at this point in your math education.

Recall that if $z$ is a complex number, with $z = a + bi$ for some real numbers $a$ and $b$, then the complex conjugate of $z$ is $\bar{z} = a - bi$. For example $i = -i$ and $2 = 2$. It satisfies

- If $z + s = r$ then $\bar{z} + \bar{s} = \bar{r}$.
- If $zs = r$ then $\bar{z}\bar{s} = \bar{r}$

Consider the 2-dimensional complex vector space $\mathbb{C}^2$. We will define an additional complex vector space $W$, which is exactly like $\mathbb{C}^2$ except with a different scalar multiplication, which we denote by $\bar{\cdot}$. Here is its formula: for any $z, r, s \in \mathbb{C}$,

$$z \bar{\cdot} \begin{bmatrix} r \\ s \end{bmatrix} = \bar{z} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \bar{z}r \\ \bar{z}s \end{bmatrix}$$

I solemnly swear that $W$ is a vector space; for this assignment you do not need to check that the axioms are all true.

Now consider two functions $T : \mathbb{C}^2 \to W$ and $\bar{T} : \mathbb{C}^2 \to W$ defined by

- $T \left( \begin{bmatrix} r \\ s \end{bmatrix} \right) = \begin{bmatrix} r \\ s \end{bmatrix}$
- $\bar{T} \left( \begin{bmatrix} r \\ s \end{bmatrix} \right) = \begin{bmatrix} r \\ s \end{bmatrix}$

Here are your tasks:

(a) Show that $T$ IS NOT a linear transformation
(b) Show that $\bar{T}$ IS a linear transformation