Instructions

1. You may use any type of calculator, but no other electronic devices during this exam.
2. You may use one page of notes, but no books or other assistance during this exam.
3. Write your Name, PID, and Section on the front of your Blue Book.
4. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.

1. Let \( f : [0, 1] \to \mathbb{R} \) be given by

\[
f(x) = \begin{cases} 
0 & \text{if } x \text{ is rational,} \\
1 & \text{if } x \text{ is irrational.}
\end{cases}
\]

Prove that \( f \) is not integrable on \([0, 1]\).

2. Let \( f : [a, b] \to \mathbb{R} \) be continuous. Use the First Fundamental Theorem to prove that

\[
\frac{d}{dx} \left[ \int_x^b f \right] = -f(x) \quad \text{for all } x \in (a, b).
\]

(Note: This result motivates the definition \( \int_c^d f = -\int_d^c f \) for \( c < d \) in \([a, b]\). You must, of course, prove the result without using this definition.)

3. Let \( f : [a, b] \to \mathbb{R} \) be given by

\[
f(x) = \begin{cases} 
0 & \text{if } a \leq x < \frac{a+b}{2}, \\
1 & \text{if } \frac{a+b}{2} \leq x \leq b.
\end{cases}
\]

Show that there is no point \( x_0 \in [a, b] \) at which \( f(x_0) = \frac{1}{b-a} \int_a^b f \). Explain why this does not contradict the Mean Value Theorem for Integrals.

4. Prove that \( 1 + \frac{1}{2}x - \frac{1}{8}x^2 < \sqrt{1+x} < 1 + \frac{1}{2}x \) for every \( x > 0 \).
5. Let \( p_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k \), the \( n \)th Taylor polynomial for \( \ln(1 + x) \).

Prove the following inequalities.

(a) \( |\ln(1 + x) - p_n(x)| \leq \frac{1}{1 + x} \cdot \frac{|x|^{n+1}}{n+1} \) if \(-1 < x \leq 0\)

(b) \( |\ln(1 + x) - p_n(x)| \leq \frac{x^{n+1}}{n+1} \) if \(0 \leq x \leq 1\)

6. For each natural number \( n \), define \( f_n : \mathbb{R} \to \mathbb{R} \) by

\[
f_n(x) = e^{-nx^2}.
\]

(a) Determine the function \( f : \mathbb{R} \to \mathbb{R} \) that \( \{f_n\} \) converges to pointwise.

(b) Prove that the convergence is not uniform.

7. Exhibit an example of a sequence of differentiable functions \( f_n : (-1, 1) \to \mathbb{R} \) that converges uniformly but for which \( f'(0) \) is unbounded (that is, \( \lim_{n \to \infty} |f'_n(0)| \) diverges to \( \infty \)).