Let's go backwards so it looks easier:

P & Q have a common factor $\iff$

$\exists$ φ, ψ, R s.t. $P = R\varphi$ & $Q = R\psi$

of degree ≥ 1 $\Rightarrow$ degree $(\varphi) <$ degree P $\deg \psi < \deg Q$

$\iff$ $P\psi = Q\varphi$ $\deg(\varphi) < m$ $\deg \psi < n$

<u>Note</u>: If $P(x,y,z)$ and $Q(x,y,z)$ are polynomials in 3 variables, we can regard them as polynomials in the variable x with coefficients in the ring $\mathbb{C}[y,z]$ of polynomials in y & z. Then $R_{P,Q}(y,z)$ will be a polynomial in y & z.

The previous proof works with coefficients in the field $\mathbb{C}(y,z)$ of rational functions in y and z. and we obtain that $\mathcal{R}_{P,Q}(y,z) \equiv 0 \iff P$ and Q have a common factor which is a polynomial in x with coefficients in $\mathbb{C}(y,z)$.

<u>Lemma 3.4</u> : Assume $P(x,y,z)$ and $Q(x,y,z)$ are nonconstant homogeneous s.t. $P(1,0,0) \neq 0$ and $Q(1,0,0) \neq 0$. Then P & Q have a nonconstant homogeneous common factor $\iff \mathcal{R}_{P,Q}(y,z) \equiv 0$.

<u>Note</u>: $P(1,0,0) \neq 0$ means $P(x,y,z) = ax^n + \cdots$ with $a \neq 0$
$n = \deg P$

<u>Proof</u>: After possibly multiplying P & Q by nonzero constants, we can write $P(x,y,z) = x^m + \cdots$

$$Q(x,y,z) = x^n + \cdots$$

From Lemma 3.3: $\mathcal{R}_{P,Q}(y,z) \equiv 0 \iff$

$\exists\ \varphi(x,y,z) \in \mathbb{C}(y,z)[x]$ with $1 \leq \deg_x \varphi$

s.t. $\varphi \mid P$ and $\varphi \mid Q$.

So $\exists$ $\psi_1(x,y,z) \in \mathbb{C}(x,y)[x]$

$\psi_2(x,y,z) \in \mathbb{C}(x,y)[x]$

s.t. $P(x,y,z) = \psi_1(x,y,z)\,\varphi(x,y,z)$

$Q(x,y,z) = \psi_2(x,y,z)\,\varphi(x,y,z)$

The denominators are polynomials in y, z.

So $\exists$ $f_1(y,z) \in \mathbb{C}[y,z]$, $f_2(y,z) \in \mathbb{C}[y,z]$

$\Psi_1(x,y,z)$, $\Psi_2(x,y,z)$, $\Phi(x,y,z) \in \mathbb{C}[x,y,z]$

s.t. $P(x,y,z)\, f_1(y,z) = \Psi_1(x,y,z)\, \Phi(x,y,z)$

$Q(x,y,z)\, f_2(y,z) = \Psi_2(x,y,z)\, \Phi(x,y,z)$

Any irreducible factor of Φ which has positive degree in x divides P and Q because it cannot divide f_1 or f_2.

<u>Lemma A.1</u>: If P is homogeneous and $\varphi | P$, then φ is homogeneous.

<u>Proof</u>: $\exists$ R s.t. $P = R\varphi$.

$$R = R_m + \cdots + R_n \qquad \left.\begin{matrix} R_i \\ \varphi_j \end{matrix}\right\} \text{ homogeneous } \neq 0$$

$$\varphi = \varphi_g + \cdots + \varphi_h \qquad \text{of degrees } i, j \text{ respectively.}$$

$$P = R\varphi = \left(\sum R_i\right)\left(\sum \varphi_j\right) = \sum_{\substack{m \leq i \leq n \\ g \leq j \leq h}} \varphi_j \cdot R_i$$

$\varphi_j \cdot R_i$ is homogeneous of degree $i+j$

$$P = \varphi_m R_g + \left(\varphi_m R_{g+1} + \varphi_{m+1} R_g\right) + \cdots + \left(\varphi_{h-1} R_n + \varphi_h R_{n-1}\right) + \varphi_h R_n$$

$\varphi_m R_g \neq 0$ hom. of degree $m+g$

$\varphi_h R_n \neq 0$ hom. " $n+h \geqslant m+g$

$\Longrightarrow$ $n+h = m+g = \deg(P)$ and $\varphi_m R_g = \varphi_h R_n$

$n \geqslant m$ & $h \geqslant g$ $\Longrightarrow$ $n = m$ & $h = g$

$\Longrightarrow$ $R = R_m$ homogeneous, $\varphi = \varphi_g$ homogeneous.

$\square$

Lemma 3.6: If $P(x) = \prod_{i=1}^{m} (x - \lambda_i)$

$Q(x) = \prod_{j=1}^{n} (x - \mu_j)$, $\lambda_i, \mu_j \in \mathbb{C}$

Then $\mathcal{R}_{P,Q} = \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (\mu_j - \lambda_i)$

In particular, $\mathcal{R}_{P,QR} = \mathcal{R}_{P,Q}\, \mathcal{R}_{P,R}$

The result also holds if $P, Q, R \in \mathbb{C}[x, y, z]$.

Proof: Consider P and Q as homogeneous polynomials in $x, \lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n$.

The entries of the Sylvester matrix are:

$$(-1)^m \prod_{i=1}^{m} \lambda_i, \ (-1)^{m-1} \sum_{j=1}^{m} \prod_{i \neq j} \lambda_i, \ \dots, \ -\sum_{i=1}^{m} \lambda_i, \ 1$$

$$(-1)^n \prod_{j=1}^{n} \mu_j, \ (-1)^{n-1} \sum_{k=1}^{n} \prod_{j \neq k} \mu_j, \ \dots, \ -\sum_{j=1}^{n} \mu_j, \ 1$$

All the coefficients are homogeneous polynomials in $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n$.

degree of the i,j entry $= \begin{cases} m+i-j & \text{if } i \leq n \\ i-j & \text{if } i > n \end{cases}$

If $r_{i,j}$ denotes the i,j entry,

$$\mathcal{R}_{P,Q} = \sum_{\sigma \in \mathfrak{S}_{m+n}} (-1)^{\operatorname{sgn}(\sigma)} \prod_{i=1}^{m+1} r_{i\sigma(i)}$$

$$\deg\left(\prod_{i=1}^{m+n} r_{i\sigma(i)}\right) = \sum_{i=1}^{m+n} \deg\left(r_{i\sigma(i)}\right)$$

$$= \sum_{i=1}^{n} (m+i-\sigma(i)) + \sum_{i=n+1}^{n+m} (i-\sigma(i))$$

$$= mn + \sum_{i=1}^{m+n} i - \sum_{i=1}^{m+n} \sigma(i) = mn$$

So $\mathcal{R}_{P,Q}$ is homogeneous of degree mn in

$\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n$