Let's go backwards as it looks easier:

$P \& Q$ have a common factor $\implies$

$\exists \, \varphi, \psi, R \text{ s.t. } P = R\varphi \, \& \, Q = R\psi$

of degree $\geq 1$ $\implies$ degree $(\varphi) < \text{degree} P$ \, deg $\psi < \text{deg} Q$

$\implies$ $P\varphi = Q\psi$ \, deg $(\varphi) < u$ \, deg $\psi < u$

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Note: If $P(x, y, z)$ and $Q(x, y, z)$ are polynomials in 3 variables, we can regard them as polynomials in the variable $u$ with coefficient in the ring $\mathbb{C}[y, z]$ of polynomials in $y \& z$. Then $R_{p, Q}(y, z)$ will be a polynomial in $y \& z$. 
The previous proof works with coefficients in the field $\mathbb{C}(y, z)$ of rational functions in $y$ and $z$, and we obtain that $\mathfrak{R}_{P, Q}(y, z) \equiv 0 \iff P$ and $Q$ have a common factor which is a polynomial in $x$ with coefficients in $\mathbb{C}(y, z)$.

**Lemma 3.4**: Assume $P(x, y, z)$ and $Q(x, y, z)$ are nonconstant homogeneous w.t. $P(1, 0, 0) \neq 0$ and $Q(1, 0, 0) \neq 0$. Then $P$ & $Q$ have a nonconstant homogeneous common factor $\iff \mathfrak{R}_{P, Q}(y, z) \equiv 0$.

**Note**: $P(1, 0, 0) \neq 0$ means $P(x, y, z) = a x^n + \cdots$ with $a \neq 0$
Proof: After possibly multiplying $P$ & $Q$ by nonzero constants, we can write

\[ P(x, y, z) = x^m + \ldots \]
\[ Q(x, y, z) = x^n + \ldots \]

From Lemma 3.3: $P, Q(x, y, z) \equiv 0 \implies$

\[ \exists \varphi(x, y, z) \in \mathbb{C}(y, z)[x] \text{ with } 1 \leq \deg_x \varphi \]

such that $\varphi \mid P$ and $\varphi \mid Q$.

So, \[ \exists \quad \varphi_1(x, y, z) \in \mathbb{C}(x, y)[x] \]
\[ \varphi_2(x, y, z) \in \mathbb{C}(x, y)[x] \]

such that

\[ P(x, y, z) = \varphi_1(x, y, z) \varphi(x, y, z) \]
\[ Q(x, y, z) = \varphi_2(x, y, z) \varphi(x, y, z) \]

The denominators are polynomials in $y, z$. 
So \( \exists f_1(y, z) \in C[y, z], f_2(y, z) \in C[y, z] \)
\[ \Phi_1(x, y, z), \Phi_2(x, y, z), \Phi(x, y, z) \in C[x, y, z] \]
\[ \text{s.t.} \quad P(x, y, z) f_1(y, z) = \Phi_1(x, y, z) \Phi(x, y, z) \]
\[ Q(x, y, z) f_2(y, z) = \Phi_2(x, y, z) \Phi(x, y, z) \]

Any irreducible factor of \( \Phi \) which has positive degree in \( x \) divides \( P \) and \( Q \) because it cannot divide \( f_1 \) or \( f_2 \).

Lemma A.1: If \( P \) is homogeneous and \( \Phi | P \), then \( \Phi \) is homogeneous.

Proof: \( \exists R \text{ s.t. } P = R \Phi \).
\[ R = R_u + \ldots + R_n \]
\[ \varphi = \varphi_g + \ldots + \varphi_h \]
\[ \{ \varphi_i \} \text{ homogeneous } \neq 0 \]

\[ P = R \varphi = (\sum R_i \varphi_i) (\sum \varphi_j) = \sum_{m \leq i \leq n} \sum_{g \leq j \leq h} \varphi_i R_i \]

\( \varphi_i R_i \) is homogeneous of degree \( i + j \)

\[ P = \varphi_u R_g + (\varphi_u R_{g+1} + \varphi_{u+1} R_g) + \ldots + (\varphi_{u-1} R_{u} + \varphi_u R_{u-1}) + \varphi_u R_n \]

\( \varphi_u R_g \neq 0 \) hom. of degree \( u + g \)

\( \varphi_u R_u \neq 0 \) hom. \( u + h \geq u + g \)

\[ \Rightarrow \quad u + h = u + g = \deg(P) \quad \text{and} \quad \varphi_u R_g = \varphi_u R_u \]

\[ u \geq u \quad \& \quad h \geq g \quad \Rightarrow \quad u = u \quad \& \quad h = g \]

\[ \Rightarrow \quad R = R_u \text{ homogeneous, } \varphi = \varphi_g \text{ homogeneous.} \]
Lemma 3.6: \[ P(x) = \prod_{i=1}^{m} (x - \lambda_i), \]
\[ Q(x) = \prod_{j=1}^{n} (x - \mu_j), \]
\[ \lambda_i, \mu_j \in \mathbb{C} \]

Then \[ R_{P,Q} = \prod_{1 \leq i \leq m, 1 \leq j \leq n} (\mu_j - \lambda_i). \]

In particular, \[ R_{P,Q} R_P = R_P Q R_P R \]

The result also holds if \[ P, Q, R \in \mathbb{C}[x, y, z] \].

Proof: Consider \( P \) and \( Q \) as homogeneous polynomials in \( x, \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n \).

The entries of the Sylvester matrix are:
\[ (-1)^{m+n} \prod_{i=1}^{m} \lambda_i, (-1)^{m+n-1} \sum_{j=1}^{n} \prod_{i \neq j}^{m} \lambda_i, \ldots, (-1)^{m+n} \sum_{i=1}^{m} \prod_{j \neq i}^{n} \lambda_i, 1 \]
\[ (-1)^{n} \prod_{j=1}^{n} \mu_j, (-1)^{n+1} \sum_{k=1}^{m} \prod_{j \neq k}^{n} \mu_j, \ldots, (-1)^{n} \sum_{j=1}^{n} \prod_{k \neq j}^{m} \mu_j, 1 \]
All the coefficients are homogeneous polynomials in $a_1, \ldots, a_m, \mu_1, \ldots, \mu_n$.

Degree of the $i, j$ entry = \[\begin{cases} \text{m} + i - j & \text{if } i \leq n \\ i - j & \text{if } i > n \end{cases}\]

If $r_{i,j}$ denote the $i,j$ entry,

\[R_{p,q} = \sum_{\sigma \in \mathcal{S}} (-1)^{s_{\sigma}(p)} \prod_{i=1}^{m} r_{i, \sigma(i)}\]

\[\deg \left( \sum_{i=1}^{m+n} r_{i, \sigma(i)} \right) = \sum_{i=1}^{m} \deg \left( r_{i, \sigma(i)} \right)\]

\[= \sum_{i=1}^{m} (m + i - \sigma(i)) + \sum_{i=m+1}^{m+n} (i - \sigma(i))\]

\[= mn + \sum_{i=1}^{m+n} i - \sum_{i=1}^{m+n} \sigma(i) = mn\]
So \( P, Q \) is homogeneous of degree \( mn \) in \\
\( a_1, \ldots, a_m, \mu_1, \ldots, \mu_n \)