Some examples:

Def: A conic is a curve of degree 2 in $\mathbb{P}^2$.

Corollary 3.12: Suppose $C \subset \mathbb{P}^2$ is an irreducible conic. Then, after a change of coordinates, C has equation

$$x^2 - yz = 0.$$

Proof: C has finitely many singular points. Choose a nonsingular point of C. After a change of coordinates, we can assume this point is $(0,1,0)$. The equation of the tangent line to C at this point is of the form $ax + by + cz = 0$, the point $(0,1,0)$ is on this line $\Rightarrow$ $b = 0$. So the equation is

$ax + cz = 0$, we can replace z with $ax + cz$

so the equation of the tangent line becomes $z=0$

The equation of C has the form:

$$P(x,y,z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2$$

$(0,1,0) \in C \Rightarrow c=0$

($z=$ equation of tangent line to C at $(0,1,0)$)

$\Rightarrow \frac{\partial P}{\partial x} = 0$ at $(0,1,0) \Rightarrow b=0$

$$\Rightarrow P = ax^2 + dxz + eyz + fz^2$$

C irreducible $\Rightarrow a \neq 0, \; e \neq 0$

divide through by a: $P(x,y,z) = x^2 + \frac{d}{a}xz + \frac{e}{a}yz + \frac{f}{a}z^2$

complete the square: $P = (x+\lambda z)^2 - z\left(\mu z + \frac{e}{a}y\right)$

replace x with $x+\lambda z$, y with $\mu z + \frac{e}{a}y$ to obtain

$$P = x^2 - yz \qquad \square$$

<u>Corollary</u>: Irreducible conics are nonsingular.

<u>Proof</u>: After a change of coordinates, the equation of the conic becomes $x^2 - yz = 0$: this defines a smooth curve.

<u>Note</u>: If C is reducible, then $P = P_1 P_2$ with P_1, P_2 of degree 1. Then $C = L_1 \cup L_2$ with

$L_i = Z(P_i)$ a line. (is nonsingular)

$\{\text{Singular points of } C\} = L_1 \cap L_2$

<u>Note</u>: An irreducible conic is bijective to $\mathbb{P}^1$:

Assume $P = Z(x^2 - yz)$.

Map
$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ (s,t) &\longmapsto (st, s^2, t^2) \end{aligned}$$
image is contained in C

Write the inverse from C to $\mathbb{P}^1$:

Suppose $(x, y, z) \in C$. If $y \neq 0$, define

$$\pi_1(x, y, z) = (y, x)$$

If $z \neq 0$, define, $\pi_2(x, y, z) = (x, z)$

If $y = 0$, $z = 0$, $x^2 - yz = 0$ forces $x = 0$, so this does not happen.

$$C = \underbrace{\{(x, y, z) \in C : y \neq 0\}}_{= C_1} \cup \underbrace{\{(x, y, z \in C, z \neq 0\}}_{= C_2}$$

$$C_1 \cap C_2 = \{(x, y, z) : y \neq 0 \ \& \ z \neq 0\}.$$

Define π on C as $\pi|_{C_1} = \pi_1$, $\pi|_{C_2} = \pi_2$.

To see that π is well-defined, we need to see that

$$\pi_1\big|_{C_1\cap C_2} = \pi_2\big|_{C_1\cap C_2}$$

Choose $(x,y,z) \in C_1 \cap C_2$ $\quad y\neq 0, z\neq 0,\ x^2 = yz \Rightarrow x\neq 0$

$\pi_1(x,y,z) = (y,x)$ $\qquad \pi_2(x,y,z) = (x,z)$

$$(y,x) = (zy, zx) = (x^2, zx) = (x,z)$$

$\square$.

<u>Another application of Weak Bézout:</u> (Prop 3.14)

Suppose C & D are two curves of degree n and meet in exactly n^2 points. Suppose $\exists\ m \in \{1,\dots,n-1\}$ and an irreducible curve E of degree m and mn points of $C\cap D$ which lie on E. Then $\exists$ curve F of degree $\leq n-m$ s.t. the remaining points of $C\cap D$ are all on F.

<u>Proof:</u> Suppose $C = \mathcal{Z}(P)$, $D = \mathcal{Z}(Q)$, $E = \mathcal{Z}(R)$

Choose $(a,b,c) \in E \setminus C \cap D$, then

$$(\lambda, \mu) := (P(a,b,c), Q(a,b,c)) \neq (0,0).$$

So $-\mu P + \lambda Q$ is a nonzero pol. of deg. n.

$$\mathcal{Z}(-\mu P + \lambda Q) \cap E \supset \{(a,b,c)\}, C \cap D \cap E$$

$$\Rightarrow \# \mathcal{Z}(-\mu P + \lambda Q) \cap E \geq mn + 1$$

So Weak Bézout $\Rightarrow$ E & $\mathcal{Z}(-\mu P + \lambda Q)$ have a common component.

$\Leftrightarrow$ $-\mu P + \lambda Q$ and R have a common factor.

E is irreducible $\Rightarrow$ R is irreducible.

$\Rightarrow$ $R \mid -\mu P + \lambda Q$ $\Rightarrow$ $\exists$ S s.t.

$$-\mu P + \lambda Q = RS, \quad \text{degree } S = n - m.$$

If $x \in C \cap D$ and $x \notin E$, then $P(x) = Q(x) = 0$

$\Rightarrow \quad R(x)\,S(x) = 0 \qquad$ but $x \notin E \Rightarrow R(x) \neq 0$

$\Rightarrow \quad S(x) = 0$

$\Rightarrow \quad C \cap D \setminus E \subset Z(S) =: F \qquad \square$

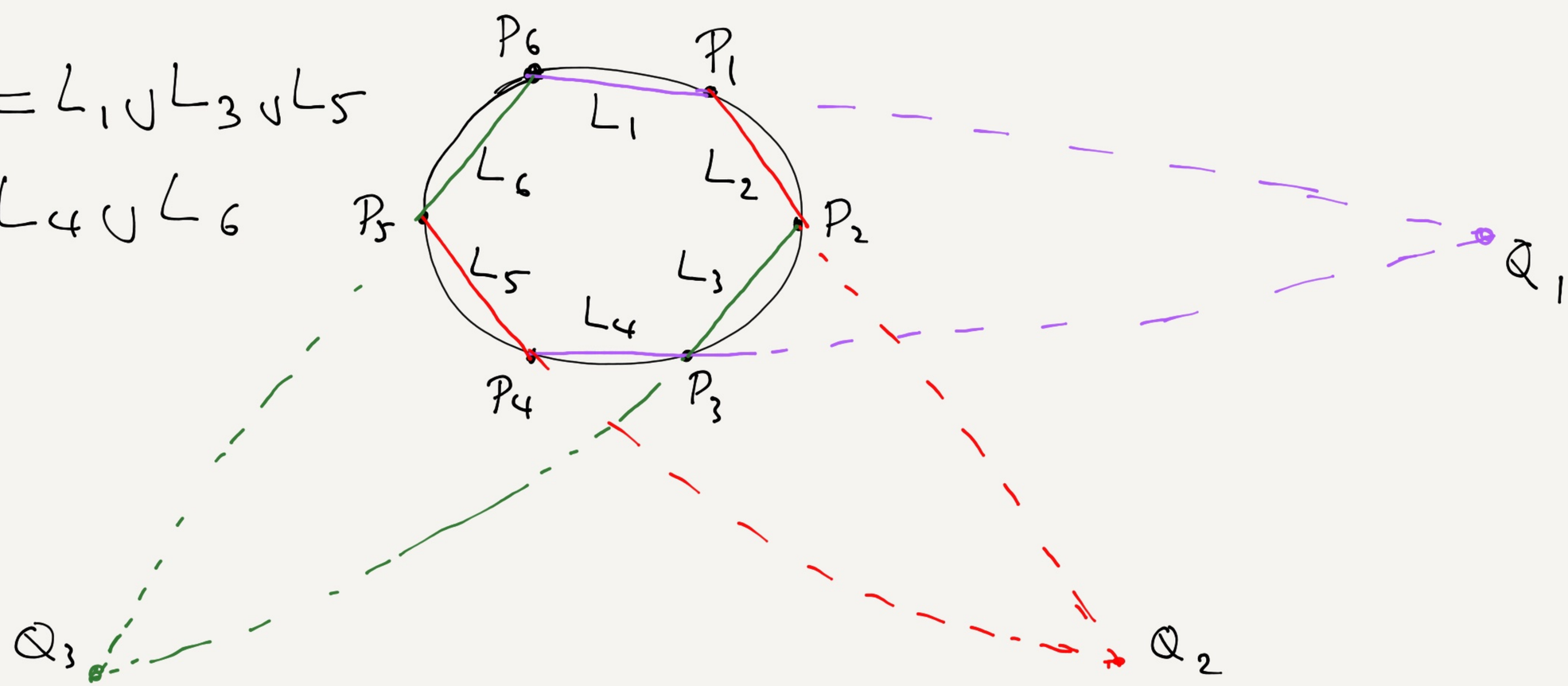
<u>Pascal's mystic hexagon:</u> (Corollary 3.15)

The pairs of opposite sides of a hexagon inscribed in a conic meet in 3 collinear points.

<u>Proof:</u>

Put $C := L_1 \cup L_3 \cup L_5$

$D := L_2 \cup L_4 \cup L_6$



$$C \cap D = \{P_1, P_2, \dots, P_6, Q_1, Q_2, Q_3\}$$

$P_1, \dots, P_6 \in$ conic $= E$

by Cor. 3.14, Q_1, Q_2, Q_3 are on a curve F of degree $\leq 1 \Rightarrow Q_1, Q_2, Q_3$ are colinear. $\square$