Some examples:

**Def:** A conic is a curve of degree $2$ in $\mathbb{P}^2$.

**Corollary 3.12:** Suppose $C \subset \mathbb{P}^2$ is an irreducible conic. Then, after a change of coordinates, $C$ has equation $x^2 - yz = 0$.

**Proof:** $C$ has finitely many singular points. Choose a nonsingular point of $C$. After a change of coordinates, we can assume this point is $(0, 1, 0)$. The equation of the tangent line to $C$ at this point is of the form $ax + by + cz = 0$. The point $(0, 1, 0)$ is on this line $\Rightarrow b = 0$. So the equation is $ax + cz = 0$, we can replace $z$ with $ax + cz$.
So the equation of the tangent line becomes \( y = 0 \)

The equation of \( C \) has the form:

\[
P(x, y, z) = ax^2 + by + cy^2 + dxz + eyz + fz^2
\]

\((0,1,0) \in C \implies c = 0\)

\((g = \text{equation of tangent line to } C \text{ at } (0,1,0))\)

\[
\Rightarrow \frac{\partial P}{\partial x} = 0 \text{ at } (0,1,0) \implies \beta = 0
\]

\[
\Rightarrow P = ax^2 + dxz + eyz + fz^2
\]

\(C \text{ irreducible } \Rightarrow a \neq 0, e \neq 0\)

Divide through by \( a \):

\[
P(x, y, z) = x^2 + \frac{d}{a}xz + \frac{e}{a}yz + \frac{f}{a}z^2
\]

Complete the square:

\[
P = (x + \lambda z)^2 - z(\mu z + \frac{e}{a}y)
\]

Replace \( x \) with \( x + \lambda z \), \( y \) with \( \mu z + \frac{e}{a}y \) to obtain

\[
P = x^2 - yz
\]

\(\Box\)
Corollary: Inedcible conics are nonsingular.

Proof: After a change of coordinates, the equation of the curve becomes \( x^2 - y^2 = 0 \); this defines a smooth curve.

Note: If \( C \) is reducible, then \( P = P_1, P_2 \) with \( P_1, P_2 \) of degree 1. Then \( C = L_1 \cup L_2 \) with \( L_i = \mathcal{L}(P_i) \) a line. (is nonsingular)

\[ \{\text{Singular point of } C\} = L_1 \cap L_2 \]

Note: An irreducible conic is bijective to \( P^1 \):

Assume \( P = \mathcal{L}(x^2 - y^2) \).

Map \( P^1 \) to \( P^2 \) via \( (s, t) \mapsto (st, s^2, t^2) \) where image is contained in \( C \).
Write the inverse from $C$ to $\mathbb{P}^1$:

Suppose $(x, y, z) \in C$. If $y \neq 0$, define

$$\tau_1(x, y, z) = (y, x)$$

If $z \neq 0$, define, $\tau_2(x, y, z) = (x, z)$

If $y = 0$, $z = 0$, $x^2 - yz = 0$ faces $x = 0$, so this does not happen.

$$C = \{(x, y, z) \in C : y \neq 0\} \cup \{(x, y, z) \in C : z \neq 0\}$$

$$C_1 \cup C_2 = \{(x, y, z) : y \neq 0 \& z \neq 0\}.$$

Define $\tau$ on $C$ as $\tau|_{C_1} = \tau_1$, $\tau|_{C_2} = \tau_2$.

To see that $\tau$ is well-defined, we need to see that
\[ \Pi_1 \mid c_1 \cap c_2 = \Pi_2 \mid c_1 \cap c_2 \]

Choose \((x, y, z) \in c_1 \cap c_2\) \(y \neq 0, z \neq 0, x = yz \Rightarrow x \neq 0\)

\[ \Pi_1 (x, y, z) = (y, x) \quad \Pi_2 (x, y, z) = (x, z) \]

\[ (y, x) = (y, y \cdot x) = (x^2, 2x) = (x, 2) \]

Another application of Weak Bézout: (Prop. 3.14)

Suppose \(C \& D\) are two curves of degree \(n\) and meet in exactly \(n^2\) points. Suppose \(E \in \{1, \ldots, n-1\}\) and an irreducible curve \(E\) of degree \(m\) and \(mn\) points of \(C \& D\) which lie on \(E\). Then \(E\) curve \(F\) of degree \(\leq n-m\) \(m\) the remaining points of \(C \& D\) are all on \(F\).
Proof: Suppose \( C = \mathbb{Z}(P) \), \( D = \mathbb{Z}(Q) \), \( E = \mathbb{Z}(R) \).

Choose \((a, b, c) \in E \setminus (C \cap D)\), then
\[
(\lambda, \mu) := (P(a, b, c), Q(a, b, c)) \neq (0, 0).
\]

So \(-\mu P + aQ\) is a nonzero pol. of deg. \(m\).

\[
\mathbb{Z}(-\mu P + aQ) \cap E \subset \{(a, b, c)\}, C \cap D \cap E
\]
\[
\Rightarrow \# \mathbb{Z}(-\mu P + aQ) \cap E \geq m + 1
\]

To Weak Bézout \(\Rightarrow\) \(E\) & \(\mathbb{Z}(-\mu P + aQ)\) have a common component.

\((=)\) \(-\mu P + aQ\) and \(R\) have a common factor.

\(E\) is irreducible \(\Rightarrow\) \(R\) is irreducible.

\[R \mid -\mu P + aQ \Rightarrow \exists S \neq t.\]

\[\mu P + aQ = RS, \text{ degree } S = n - m.\]

If \(x \in C \cap D\) and \(x \notin E\), then \(P(x) = Q(x) = 0\).
\[ \Rightarrow R(x) \cap S(x) = \emptyset \] \hspace{1cm} \text{but } x \notin E \Rightarrow R(x) = 0

\[ \Rightarrow \bigcap_{x \in E} \emptyset = \emptyset \]

\[ \Rightarrow C \cap D \setminus E \subset \emptyset(S) = 0 \]

\[ \square \]

\textbf{Pascal's mystic hexagon: (Corollary 3.15)}

The pairs of opposite sides of a hexagon circumscribed about a conic meet in 3 collinear points.

\textbf{Proof:}

\textbf{Part 1:}

\[ C := L_1 \cup L_3 \cup L_5 \]

\[ D := L_2 \cup L_4 \cup L_6 \]

\textbf{Part 2:}

\[ P_6 \]

\[ P_1 \]

\[ P_2 \]

\[ P_3 \]

\[ P_4 \]

\[ P_5 \]

\[ L_1 \]

\[ L_2 \]

\[ L_3 \]

\[ L_4 \]

\[ L_5 \]

\[ L_6 \]

\textbf{Q_1, Q_2, Q_3}
\[ C \cap D = \{ P_1, P_2, \ldots, P_6, Q_1, Q_2, Q_3 \} \]

\[ P_1, \ldots, P_6 \in \text{conic} = E \]

by Cor. 3.14, \( Q_1, Q_2, Q_3 \) are on a curve \( F \) of degree \( \leq 1 \) \( \Rightarrow \) \( Q_1, Q_2, Q_3 \) are colinear.