

We want to define intersection multiplicities.

Recall that for two homogeneous polynomials  $P(x, y, z)$ ,  $Q(x, y, z)$ ,  $R_{P,Q}(y, z)$  is homogeneous of degree  $m+n$  in  $y, z$ , where  $m = \deg P$ ,  $n = \deg Q$ . (Lemma 3.7)

$$R_{P,Q}(b, c) = R_{P(x, b, c), Q(x, b, c)} \\ = 0 \iff P(x, b, c) \& Q(x, b, c) \text{ have a common root.} \quad (\text{Lemma 3.3})$$

$$\Leftrightarrow \exists a \text{ s.t. } P(a, b, c) = Q(a, b, c) = 0$$

also  $bz - cy \mid R_{P,Q}(y, z) \Leftrightarrow R_{P,Q}(b, c) = 0$ .

(A) (Lemma 3.4) We need  $(1, 0, 0) \notin C \cup D$  so that we have  $R_{P,Q}(y, z) = 0 \Leftrightarrow P \& Q \text{ have a nonconstant common factor.}$

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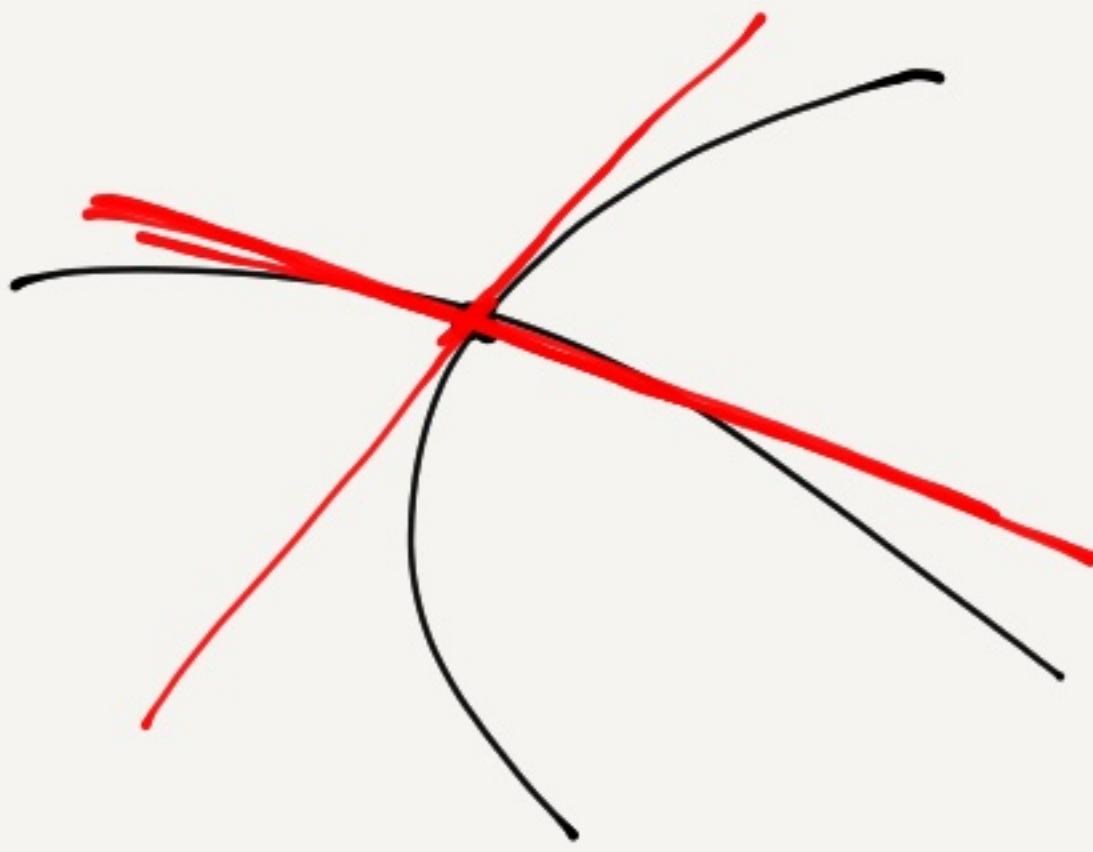
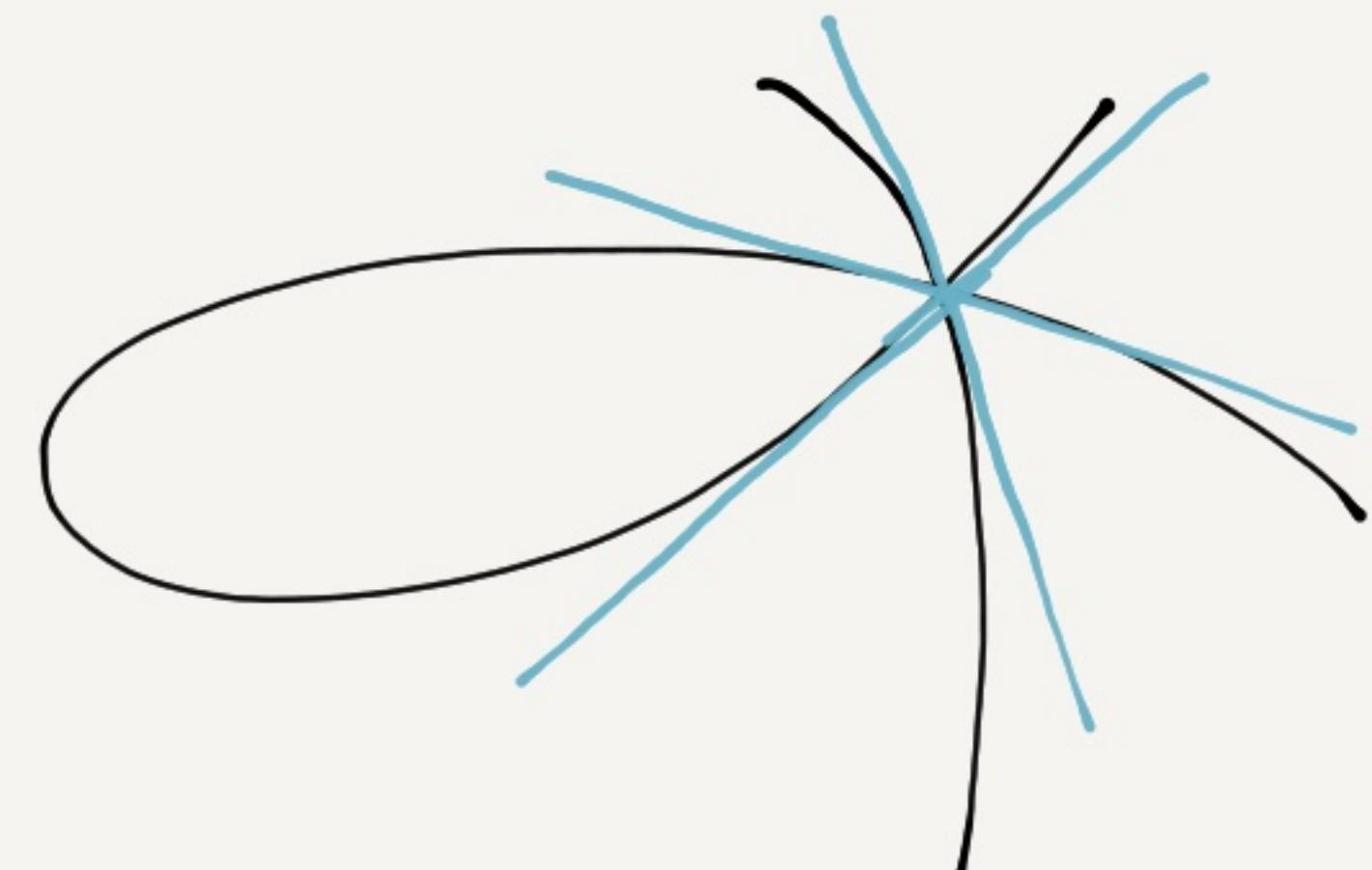
Recall that for  $(bz - cy)$  and  $(b'z - c'y)$  to be nonproportional for points  $(a, b, c), (a', b', c') \in C \cap D$ , we need

$(1, 0, 0)$  and  $(a, b, c), (a', b', c)$  to be noncollinear. So we need

(B)  $(1, 0, 0) \notin$  any line through two points of  $C \cap D$ .

We also need

(C)  $(1, 0, 0) \notin$  any line in a tangent cone to  $C \cap D$  at a point of  $C \cap D$ .



We want this condition because we want to take limits as two points of  $C \cap D$  come together

We can always find a point  $\notin C \cup D$ , all lines through 2 points of  $C \cap D$  all tangent lines at points of  $C \cap D$

then we make a change of coordinates so this point becomes  $(1, 0, 0)$ .

Def: For such a choice of coordinates, we define

(ABC)  $I_f(c, d) = I_f(P, Q) := \text{largest } k \text{ s.t.}$   
 $(b\beta - c\gamma)^k \mid R_{P,Q} \quad \text{where } f = (a, b, c).$

We need to show this is coordinate independent. So we isolate a finite number of properties of intersection multiplicities which completely determine them.

Theorem 3.18:  $\exists ! I_f(P, Q) \in \mathbb{Z} \cup \{\infty\}$

defined for all  $f \in \mathbb{P}^2$  and  $P, Q$  hom. pt. in  $(x, y, z)$

satisfying the following properties:

(i)  $I_f(P, Q) = I_f(Q, P)$

(ii)  $I_f(P, Q) = \infty \Leftrightarrow f \text{ is a zero of a common factor of } P \text{ and } Q$

otherwise  $I_f(P, Q) \geq 0$ .

(iii)  $I_P(P, Q) = 0 \iff P \notin Z(P) \cap Z(Q)$

(iv) If  $P, Q$  are linear, non-proportional, then

$$I_P(P, Q) = 1 \text{ for } \{P\} = Z(P) \cap Z(Q)$$

(v) If  $P = P_1 P_2$ , then  $I_P(P, Q) = I_P(P_1, Q) + I_P(P_2, Q)$

(vi)  $\forall$  homogeneous mol.  $R$ :  $I_P(P, RP + Q) = I_P(P, Q)$   
(assuming  $\deg P \leq \deg Q$ )

Proof: Define  $I_P(P, Q)$  as in Def. ABC. We will

show that it satisfies properties (i), ..., (vi).

For (i):  $R_{Q,P} = \pm R_{P,Q}$  so they have the same  
factors.  $\Rightarrow$  (i)

For (ii): By Lemma 3.4,  $R_{P,Q} = 0 \iff P \& Q \text{ have a common factor.}$   
 $(P(1,0,0) \neq 0 \neq Q(1,0,0))$

$$\downarrow k = \infty \Rightarrow I_P(P, Q) = \infty.$$

$I_P(P, Q) \geq 0$  clear..

For (iii):  $k > 0$  means  $b_3 - c_3 \neq \mathcal{R}_{P,Q}$

which means  $\mathcal{R}_{P,Q}(b,c) = 0 \quad (\Rightarrow \exists a \text{ s.t. } P(a,b,c) = Q(a,b,c) = 0)$

$(\Rightarrow) \quad (a,b,c) \in Z(P) \cap Z(Q).$

For (iv): Compute:  $P = ax + by + cz \quad Q = \alpha x + \beta y + \gamma z$

$$\mathcal{R}_{P,Q} = \begin{vmatrix} by+cz & a \\ \beta y+\gamma z & \alpha \end{vmatrix} = \alpha(by+cz) - a(\beta y+\gamma z) \\ = (\alpha b - a\beta)y + (\alpha c - a\gamma)z$$

one linear factor with mult. 1  $\Rightarrow I_F(P,Q) = 1$

for  $\mu = \text{the point}$   
of intersection of  
 $Z(P) \& Z(Q)$ .

For (v): By Lemma 3.6:  $\mathcal{R}_{P_1, P_2, Q} = \mathcal{R}_{P_1, Q} \mathcal{R}_{P_2, Q}$

$\Rightarrow \text{if } k_1 = I_F(P_1, Q), k_2 = I_F(P_2, Q) \Rightarrow I_F(P, Q) = k_1 + k_2.$

Fin (v): Go back to the definition of  $\mathcal{R}_{P,Q}$ :

Write:  $\mathcal{R}_{P,Q}(y,z) = \det \left( r_{ij}(y,z) \right)_{\substack{i \leq i, j \leq m+n}}$

$$R(x,y,z) = p_0(y,z) + p_1(y,z)x + \cdots + p_{n-m}(y,z)x^{n-m}$$

$$RP + Q \text{-homog.} \Rightarrow \deg Q = \deg R + \deg P$$

If  $\mathcal{R}_{P,RP+Q} = \det \left( s_{ij}(y,z) \right)_{\substack{i \leq i, j \leq m+n}}$

then  $s_{ij} = \begin{cases} r_{ij} & \text{if } i \leq n \\ r_{ij} + \sum_{k=i-n}^{i-m} p_{i-m-k} r_{kj} & \text{if } i > n \end{cases}$

We've done some row operations  $\Rightarrow \det$  doesn't change.

$$\Rightarrow \mathcal{R}_{P,Q} = \mathcal{R}_{P, PR+Q}.$$

Done with existence.

Uniqueness: We show that conditions (i), ..., (vi) completely determine  $I_p(P, Q)$ , i.e., it can be computed for any  $p, P, Q$ , knowing just (i), ..., (vi).

From (ii) we know when  $I_p(P, Q)$  is  $\infty$ .

So we can assume  $I_p(P, Q)$  is finite, again by (ii).  $\geq 0$ .

From (i) and (v) we can assume  $P$  &  $Q$  are irreducible.

Now we use induction.

Induction hypothesis: If  $I_p(P, Q) < k$ , then  $I_p(P, Q)$  can be computed by just knowing (i), ..., (vi)

Initial step:  $k=1$  : we know when  $I_p(P, Q)=0$  by (iii)

Assume  $k \geq 2$ .  $m = \deg P$ ,  $n = \deg Q$ ,  $p \in \mathbb{P}^2$

assume  $p = (0, 0, 1)$

Write:  $P(x, y, z) = P(x, 0, z) + y R(x, y, z)$

$Q(x, y, z) = Q(x, 0, z) + y S(x, y, z)$

Put  $r := \deg_x P(x, 0, 1)$ ,  $s := \deg_x Q(x, 0, 1)$   
 switch  $P$  &  $Q$  if necessary, so that  $r \leq s$ .

also divide by nonzero scalars if necessary so that

$P(x, 0, 1)$  &  $Q(x, 0, 1)$  are monic.

define  $T(x, y, z) := \beta^{m+s-r} Q(x, y, z) - x^{s-r} y^r P(x, y, z)$

then  $T(x, 0, 1) = Q(x, 0, 1) - x^{s-r} P(x, 0, 1)$

and  $\deg_x T(x, 0, 1) < r$

$$I_p(P, T) \underset{\text{by (vi)}}{=} I_p(P, \beta^{m+s-r} Q)$$

$$\underset{\text{by (vii)}}{=} I_p(P, Q) + (m+s-r) I_p(P, z) \underset{\text{by (iii)}}{=}$$

$$= I_p(P, Q)$$

replace  $Q$  with  $T$  and repeat. if  $r \leq s-1$ , if not  
 switch  $P$  &  $Q$  and repeat.

We can keep doing this until  $n=0$ .

Now we have  $P(x,y,z) = \lambda R^m + y R(x,y,z)$   
since  $P(0,0,1) = 0$ , we have  $\lambda = 0$