As we saw before, if \( I_p(P, x) \geq 3 \), then
\[ y^3 | P(0, y, z) \text{ and } \mathcal{H}_p(x) = 0. \]

The converse: ?

Assume \( \mathcal{H}_p(x) = 0 \), what can we say?

As before, assume \( P = (0, 0, 1) \) and \( L = \{ x = 0 \} \) is tangent to \( C \) at \( P \), so that \( \frac{\partial P}{\partial y}(0, 0, 1) = \frac{\partial P}{\partial z}(0, 0, 1) = 0 \)
and we have \( R_{P, x} = P(0, y, z) \) & \( y^2 | P(0, y, z) \).

Lemma 3.30: Suppose \( m = \deg P > 1 \). Then

\[ 2 \cdot \mathcal{H}_P(x, y, z) = (m-1)^2 \left[ \begin{array}{ccc}
P_{xx} & P_{xy} & P_x \\
P_{yx} & P_{yy} & P_y \\
P_x & P_y & \frac{mP}{m-1}
\end{array} \right] \]

where \( P_{xy} = \frac{\partial^2 P}{\partial x \partial y} \) etc.
Proof: Recall Euler's relation:

\[ m \mathbf{P} = x \mathbf{P}_x + y \mathbf{P}_y + z \mathbf{P}_z \]

\[(m-1) \mathbf{P}_x = x \mathbf{P}_{xx} + y \mathbf{P}_{xy} + z \mathbf{P}_{xz} \]
\[(m-1) \mathbf{P}_y = x \mathbf{P}_{yx} + y \mathbf{P}_{yy} + z \mathbf{P}_{yz} \]
\[(m-1) \mathbf{P}_z = x \mathbf{P}_{zx} + y \mathbf{P}_{zy} + z \mathbf{P}_{zz} \]

Multiply the last column by 3 and add \(x\) times the first column plus \(y\) times the second column, then do the same with the rows to obtain the result. \(\square\)

Apply the lemma: At \(p = (0, 0, 1)\) we have

\[ 0 = \mathcal{H}_P(p) = (m-1)^2 \begin{vmatrix} p_{xx}(p) & p_{xy}(p) & p_{x}(p) \\ p_{yx}(p) & p_{yy}(p) & 0 \\ p_{zx}(p) & p_{zy}(p) & 0 \end{vmatrix} = -(m-1)^2 p_{x} p_{y} \]
So \( \{ \) either \( P_x(p) = 0 \) which means \( p \) is a singular point of \( C \)

or \( P_{yy}(p) = 0 \) which means \( y^3 \mid P(0,y;3) \)

So either \( p \) is a singular point of \( C \)

or \( I_p(C,L) \geq 3 \).

So we have the conclusion: \( \exists \) line \( L \) s.t.

\( I_p(C,L) \geq 3 \implies H_p(p) = 0 \).

i.e., \( p \) is a point of inflexion \( \implies H_p(p) = 0 \)

Remark: Euler's relations for \( P_x, P_y, P_y \) show that if \( p \) is singular on \( C \), we have \( H_p(p) = 0 \).
So we can rephrase the conclusion:

\[ C \cap \mathcal{Z}(H_p) = \{ \text{nonsingular points of inflection} \} \cup \{ \text{singular points} \}. \]

First application:

**Corollary 3.34**: Suppose \( C \) is a nonsingular cubic. Then there coordinates \( \mathcal{F} \) in which the equation of \( C \) is

\[ y^2 z = x(x - 3)(x - 2) \]

for some \( \lambda \in \mathbb{C} \setminus \{0, 1\} \).

**Proof**: By the above \( C \cap \mathcal{Z}(H_p) = \{ \text{points of inflection} \} \).

By Bézout \( C \cap \mathcal{Z}(H_p) \neq \emptyset \) because \( \deg C = 3 \) and \( \deg H_p = 3(m-2) = 3 \).

So \( C \) has at least one point of inflection.

Suppose \( p = (0, 1, 0) \) is a point of inflection of \( C \) and \( \{ z = 0 \} \) is the tangent line to \( C \) at \( p \).
Write \( P(x, y, z) = P(x, 0, z) + y Q(x, y, z) \)

restrict to the line \( z = 0 \): \( P(x, y, 0) = P(x, 0, 0) + y Q(x, y, 0) \)

because \( C \cap L = \{ p \} \) with multiplicity 3 up to a scalar.

\[ \Rightarrow Q(x, y, 0) = 0 \Rightarrow y = Q(x, y, 3) \]

\[ \Rightarrow \exists \alpha, \beta, r \in \mathbb{C} \text{ s.t. } Q(x, y, 3) = \beta (\alpha x + \beta y + \gamma z) \]

Recap: \( P(x, y, z) = P(x, 0, z) + y \beta (\alpha x + \beta y + \gamma z) \)

\[ 0 = \frac{\partial P}{\partial z} (0, 1, 0) = \beta \]

replace \( y \) with \( y_1 := y + \frac{\alpha x + \beta y}{\gamma} \)

\[ \Rightarrow P(x, y_1, 3) = P(x, 0, 3) + \beta y^2 \gamma \]

\( P(x, 0, z) \) is homogeneous of degree 3 in \((x, z)\) so it is a
product of 3 linear factors, after a change of coordinates, we can assume one of the factors is $-x$:

$$P(x, y, z) = -x(x-a_3)(x-b_3) + y^2 z \quad \text{(replace } z \text{ with } b_3)$$

Another change of coordinates: replace $z$ with $az$.

$$\Rightarrow P(x, y, z) = -x(x-az)(x-bz) + y^2 z$$

Note: the three linear factors are not proportional: $(a \neq 0)$

If they were, $C$ would be singular.

So $A \in C \setminus \{0, \}$. □