MATH 106

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December 13 2019: Practice problems for the final exam

Please explain or prove all your assertions and show your work. Please state all the definitions, propositions, theorems, lemmas that you use precisely.

Please make sure to review all definitions, statements of theorems, proofs done in class, practice problems for the midterms and homework problems.

These problems are in addition to the practice problems for the midterms and to the midterms.

Good luck!

(1) Give two different definitions for a homogeneous polynomial and prove that they are equivalent.

(2) Give the definition of a projective transformation of $\mathbb{P}^n$.

(3) Give the definition of a hyperplane in $\mathbb{P}^n$.

(4) Suppose we are given $n+2$ distinct points $q_0, \ldots, q_{n+1}$ of $\mathbb{P}^n$, no $n+1$ of which lie on a hyperplane. Prove that there exists a unique projective transformation $f : \mathbb{P}^n \to \mathbb{P}^n$ such that $f(q_i)$ is the point all of whose homogeneous coordinates are 0 except the one in the $(i+1)$-st spot and $f(q_{n+1}) = (1, \ldots, 1)$.

(5) Prove that if $P(x) = \prod_{i=1}^{m}(x - \lambda_i)$ and $Q(x) = \prod_{j=1}^{n}(x - \mu_j)$, then

$$\mathcal{R}_{P,Q} = \prod_{1 \le i \le m, 1 \le j \le n} (\mu_j - \lambda_i).$$

(6) Without using Bézout's theorem, prove that any two projective curves meet in at least 1 point.

(7) Give an example of two affine curves that do not meet.

(8) Let p be a point of $\mathbb{P}^2$ and C and D projective plane curves of respective multiplicities s and t at p. Prove that $I_p(C, D) \ge st$.

(9) Give a **complete** proof of the following. Given a nonsingular projective plane cubic C and a point of inflection p_0 on C, there exists a unique abelian group structure on C such that $p_0 = 0$ and three points p, q, r of C sum to 0 if and only if they are collinear. Here we allow the points p, q, r to coincide, so collinear means the points with given multiplicities form the intersection of C with a line.