MATH 106
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(1) (25 points)
(a) Give the definition of \( n \)-dimensional projective space over an arbitrary field \( k \).
(b) Give the definition of the resultant of two homogeneous polynomials in \( n \) variables \( x_1, \ldots, x_n \) with respect to the variable \( x_n \).
(c) Give the definition of the multiplicity at a point of a projective plane curve.
(d) Give the definition of the tangent cone at a point of an affine plane curve.

Solution:
(a) \( n \)-dimensional projective space is the set of lines through the origin in \( k^{n+1} \).
(b) Write
\[
P(x_1, \ldots, x_n) = a_0(x_1, \ldots, x_{n-1}) + \ldots + a_l(x_1, \ldots, x_{n-1})x_n^l,
\]
\[
Q(x_1, \ldots, x_n) = b_0(x_1, \ldots, x_{n-1}) + \ldots + b_m(x_1, \ldots, x_{n-1})x_n^m,
\]
then the resultant is the determinant of the following \((l + m) \times (l + m)\) matrix
\[
\begin{pmatrix}
a_0 & a_1 & \ldots & a_l & 0 & 0 & \ldots & 0 \\
0 & a_0 & \ldots & a_l & a_l & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & a_0 & a_1 & \ldots & a_l \\
b_0 & b_1 & \ldots & b_m & 0 & \ldots & 0 \\
0 & b_0 & \ldots & b_m & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & b_0 & b_1 & \ldots & b_m
\end{pmatrix}
\]
(c) Suppose the curve \( C \) is the zero set of the homogeneous polynomial \( P(x, y, z) \). The multiplicity of \( C \) at a point \((a, b, c) \in C\) is the positive integer \( m \) such that there exists non-negative integers \( i_1, i_2, i_3 \) such that \( i_1 + i_2 + i_3 = m \) and \( \frac{\partial^m P}{(\partial x)^{i_1} (\partial y)^{i_2} (\partial z)^{i_3}}(a, b, c) \neq 0 \) and, for all non-negative integers \( i_1, i_2, i_3 \) such that \( i_1 + i_2 + i_3 < m \), we have \( \frac{\partial^m P}{(\partial x)^{i_1} (\partial y)^{i_2} (\partial z)^{i_3}}(a, b, c) = 0 \).
(d) Suppose \( C \) is the zero set of the polynomial \( f(x, y) \). At a point \((a, b) \in \mathbb{C}^2\), we can write the Taylor expansion of \( f \):

\[
f(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + \ldots + \sum_{i_1 + i_2 = m}^{\partial^m f \over \partial x^{i_1} \partial y^{i_2}} (x - a)^{i_1} (y - b)^{i_2},
\]

where \( m \) is the degree of \( f \). Then the tangent cone to \( C \) is the zero set of the lowest degree non-zero term above.

(2) (25 points)

(a) Prove that a homogeneous polynomial in two variables with complex coefficients is a product of linear factors.

(b) Prove that the resultant in \( x \) of two homogeneous polynomials in \((x, y, z)\) of degrees \( m \) and \( n \) is either zero or homogeneous of degree \( mn \). Deduce that the resultant is either \( 0 \) or the product of \( mn \) linear factors.

Solution:

(a) If \( y^d \) is the highest power of \( y \) appearing in a homogeneous polynomial \( P(x, y) \), we can write \( P(x, y) = y^d f \left( \frac{x}{y} \right) \), where \( f \) is the polynomial in one variable with the same coefficients as \( P \). By the fundamental theorem of algebra, the polynomial \( f \) is a product of polynomials of degree 1:

\[
P(x, y) = y^d f \left( \frac{x}{y} \right) = y^d \prod_i \left( a_i \frac{x}{y} - b_i \right) = \prod_i \left( a_i x - b_i y \right).
\]

(b) If \( r_{i,j} \) denotes the entry of the Sylvester matrix on row \( i \) and column \( j \), then the resultant of \( P \) and \( Q \) is

\[
R_{P,Q} = \sum_{\sigma \in \mathfrak{S}_{mn}} \text{sgn}(\sigma) \prod_i r_{i,\sigma(i)}
\]

where \( \text{sgn}(\sigma) \) is the signature of \( \sigma \) as a permutation. The entries \( r_{i,j} \) are the coefficients of \( P \) and \( Q \) as polynomials in \( x \), so they are homogeneous polynomials in \((y, z)\). The degrees \( d_{i,j} \) of the \( r_{i,j} \) as homogeneous polynomials in \((y, z)\) are given by

\[
d_{i,j} = \begin{cases} 
m + i - j & \text{if } i \leq n \\
i - j & \text{if } i > n \end{cases}
\]

So the degree of \( \prod_i r_{i,\sigma(i)} \) is

\[
\sum_{i=1}^{n} (m + i - \sigma(i)) + \sum_{i=n+1}^{n+m} i - \sigma(i) = mn + \sum_{i=1}^{m+n} i - \sum_{i=1}^{m+n} \sigma(i) = mn.
\]
So every term of \( R_{P,Q} \) is a homogeneous polynomial of degree \( mn \) in \((y,z)\), hence the same holds for \( R_{P,Q} \).

(3) (26 points)

(a) Prove that any two projective plane curves meet in at least one point.

(b) Deduce that any reducible projective plane curve is singular.

Solution:

(a) Suppose that \( C \) is the zero set of the homogeneous polynomial \( P(x, y, z) \) of degree \( m \geq 1 \), and \( D \) is the zero set of the homogeneous polynomial \( Q(x, y, z) \) of degree \( n \geq 1 \). Then, by the previous question, the resultant \( R_{P,Q} \) is either zero or a product of \( mn \geq 1 \) linear factors:

\[
R_{P,Q}(y, z) = \prod_{i=1}^{mn} (c_iy - b_iz).
\]

In either case, we have \( R_{P,Q}(b_1, c_1) = 0 \). We know that, for two polynomials \( A, B \) in one variable, \( R_{A,B} = 0 \) if and only if \( A \) and \( B \) have a common root. Since \( R_{P,Q}(b_1, c_1) = R_P(x, b_1, c_1), Q(x, b_1, c_1) \), we deduce that the polynomials in one variable \( P(x, b_1, c_1), Q(x, b_1, c_1) \) have a common root. So there exists \( a_1 \in \mathbb{C} \) such that \( P(a_1, b_1, c_1) = Q(a_1, b_1, c_1) = 0 \). This means \( C \) and \( D \) have at least one point in common.

(b) A plane curve is reducible when it is defined by a reducible polynomial (with no repeated factors), say \( P_1P_2 \). So \( C = C_1 \cup C_2 \) where \( C_i \) is the zero set of \( P_i \). By part (a), the two curves have at least one point in common, say \((a, b, c)\). At such a point, all the partials of \( P \) are zero. For instance:

\[
\frac{\partial P}{\partial x}(a, b, c) = \frac{\partial P_1P_2}{\partial x}(a, b, c) = P_2(a, b, c) \frac{\partial P_1}{\partial x}(a, b, c) + P_1(a, b, c) \frac{\partial P_2}{\partial x}(a, b, c) = 0.
\]

(4) (24 points) Find the singular points of the following projective plane curve. Determine their multiplicities and tangent cones.

\[
f(x, y, z) = x^3yz - x^5 - y^5 = 0.
\]

Solution:

To find the singular points, we set all partial derivatives equal to 0:

\[
\frac{\partial f}{\partial z} = x^3y = 0
\]
means either $x = 0$ or $y = 0$. Next, if $x = 0$, then

$$\frac{\partial f}{\partial y} = x^3 z - 5y^4 = 0$$

implies that $y = 0$. Similarly, if $y = 0$, then

$$\frac{\partial f}{\partial x} = 3x^2 y z - 5x^4 = 0$$

implies $x = 0$. Hence the only singular point is $(0, 0, 1)$. To find the tangent cone, we look in the affine plane $z = 1$:

$$f(x, y, 1) = x^3 y - x^5 - y^5.$$  

From the above equation it is clear that the first nonzero homogeneous piece of the Taylor expansion of $f(x, y, 1)$ at the point $(0, 0)$ is $x^3 y$. So the multiplicity of $(0, 0, 1)$ on $C$ is 3 and the tangent cone at $(0, 0, 1)$ is the union of the two lines of equations $x = 0$ and $y = 0$.  