

Solve equations:

$$x + 3 = 0.$$

So need negative numbers.

→ \mathbb{Z} .

$$2x - 3 = 0 \quad x = \frac{3}{2}$$

→ \mathbb{Q} .

solve: $x^2 = 2$

solution is $\pm\sqrt{2}$. not rational.

→ irrational numbers.

Algebraic numbers are solutions of polynomial equations

Not algebraic = transcendental.

e.g.: π , \underline{e} , ...

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

$$1 + x + \frac{x^2}{2} + \dots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$$

$$\mathbb{R} = \left\{ \text{limits of sequences of rational numbers} \right\}$$

$$x^2 + 1 = 0 \quad \text{has no solution in } \mathbb{R}.$$

has a solution in \mathbb{C} .

$$i^2 = -1 \quad (-i)^2 = -1$$

$$\mathbb{C} = \left\{ x + iy \mid \begin{array}{l} x \in \mathbb{R} \\ y \in \mathbb{R} \end{array} \right\}$$

Add, subtract, multiply, divide the usual way, remember $i^2 = -1$

$$1 - i + 3 - 2i = 4 - 3i$$

$$1 - i - (3 - 2i) = -2 + i$$

$$(1-i)(3-2i) = 3 - 2i - 3i + 2i^2$$

$$= 3 - 5i - 2 = 1 - 5i$$

$$\frac{1-i}{3-2i} = ? \quad (a-b)(a+b) =$$

$$(x+iy)(x-iy) = x^2 - (iy)^2$$

$$= x^2 - i^2 y^2$$

$$= x^2 + y^2$$

$$\Rightarrow (x+iy)^{-1} = \frac{x-iy}{x^2+y^2}$$

$$\frac{1-i}{3-2i} = (1-i)(3-2i)^{-1}$$

$$= (1-i) \frac{3+2i}{9+4}$$

$$= \frac{1}{13} (1-i)(3+2i)$$

$$= \frac{1}{13} (5-i) = \frac{5}{13} - i\frac{1}{13}$$

The conjugate of $x+iy = z$

is $\bar{z} = x - iy$.

$$x := \operatorname{Re}(z) \quad y := \operatorname{Im}(z).$$

$$z \leftrightarrow \text{vector in } \mathbb{R}^2$$

$$z = x + iy \leftrightarrow (x, y).$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1, y_1) + (x_2, y_2) =$$

$$(x_1 + x_2, y_1 + y_2)$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1 x_2 - y_1 y_2)$$

$$+ i(x_1 y_2 + x_2 y_1)$$

$$(x_1, y_1)(x_2, y_2) =$$

$$(x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

$$z = x + iy \quad z^{-1} = \frac{x - iy}{x^2 + y^2}$$

$$x^2 + y^2 = |z|^2$$

$|z|$ = modulus of z .

$$= \sqrt{x^2 + y^2}$$

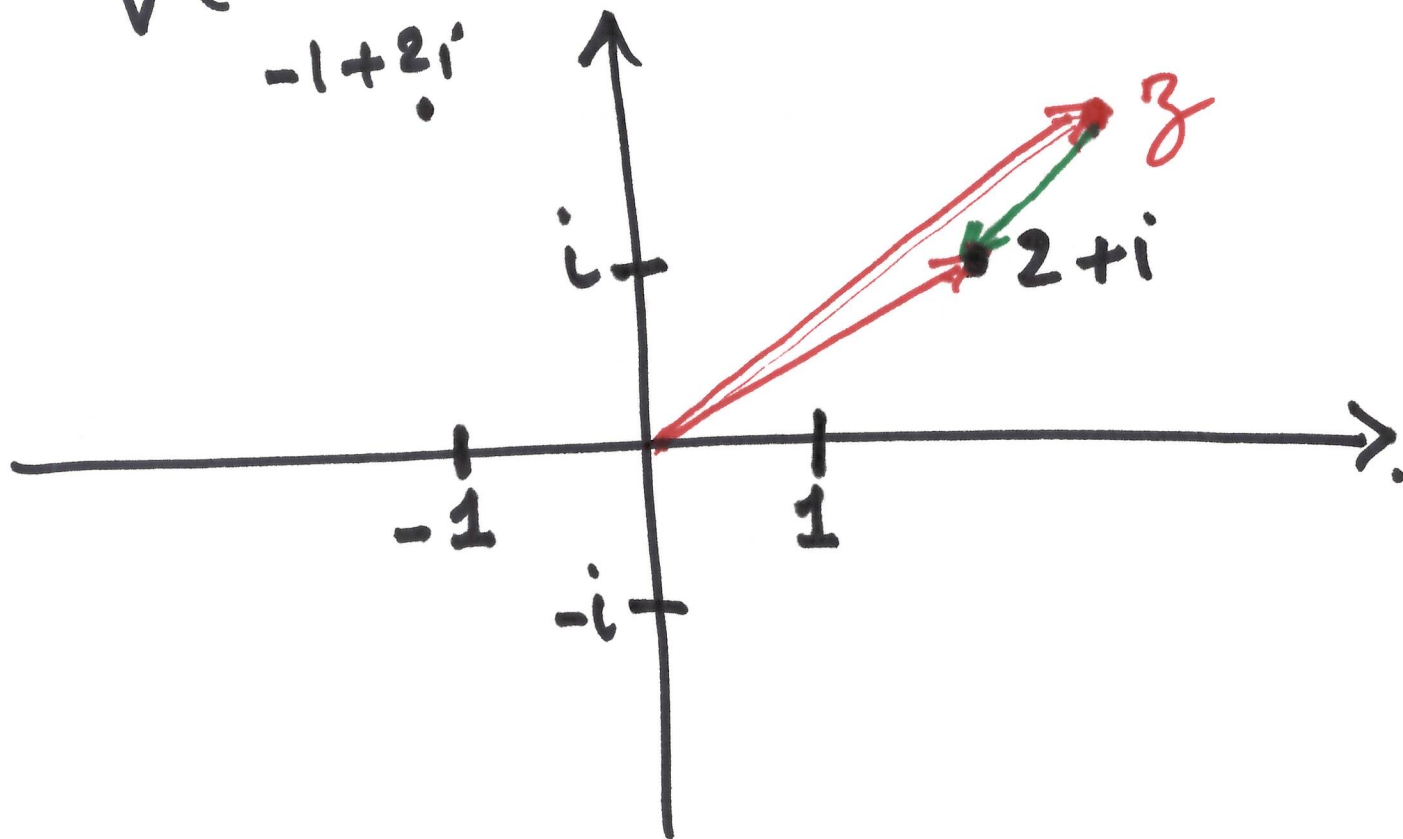
$$z_1 = x_1 + iy_1 \quad z_2 = x_2 + iy_2$$

$|z_1 - z_2|$ = length of vector

$$(x_1 - x_2, y_1 - y_2)$$

$$\rightarrow \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$-1 + 2i$$



$z \in$ circle of radius 1
centered at 0

$$\Leftrightarrow |z| = 1$$

circle of radius 1 centered at

$$2+i: |z - (2+i)| = 1$$

Note: $z \bar{z} = |z|^2.$

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

if $|z| = 1$, then $z^{-1} = \bar{z}.$

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

$$f: \mathbb{C} \longrightarrow \mathbb{C}$$

Analytic functions: they have
power series expansions

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

radius of convergence = 1.

interval of convergence: $]-1, 1[$.

function blows up at 1.

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

radius of convergence = 1

function blows up at -1.

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

radius of convergence = 1.

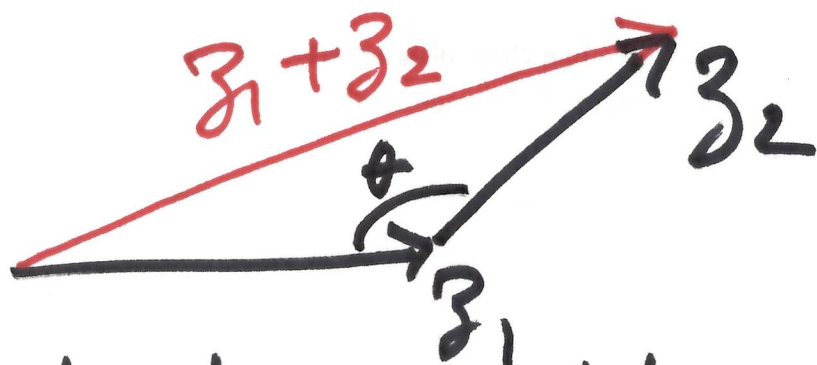
$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

blows up at $\pm i$:

radius of convergence = 1

Triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$



length of one side of a triangle
 \leq the sum of the lengths
of the other two sides.

Another way of writing it:

$$w_1 = z_1 + z_2 \quad w_2 = z_2$$

$$|w_1| \leq |w_1 - w_2| + |w_2|$$

$$|w_1| - |w_2| \leq |w_1 - w_2|$$

Similarly

$$|w_2| - |w_1| \leq |w_2 - w_1|$$

$$\text{So } ||w_1| - |w_2|| \leq |w_1 - w_2|$$

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos\theta$$

Law of cosines.

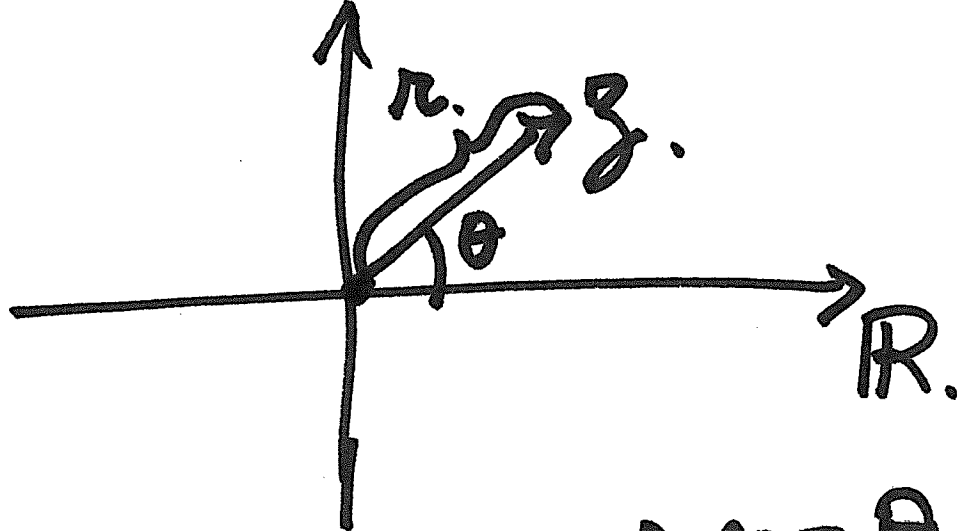
$$|z_1 + z_2|^2 \leq \cancel{18} |z_1|^2 + |z_2|^2 + 2 |z_1| |z_2|$$

$$\Rightarrow \left(|z_1| + |z_2| \right)^2.$$

$$|z_1 + z_2| \leq \underline{|z_1|} + |z_2|$$

~~Taking~~ $f(x) = \sqrt{x}$ is an increasing function of a real variable.

Polar coordinates:



$$z = x + iy$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r > 0$$

$$r = |z| \text{ modulus.}$$

$$\theta = \text{Arg}(z)$$

principal
the argument
of z .

$$-\pi \leq \theta \leq \pi$$

$$r = |z| = \sqrt{1+4} = \sqrt{5}$$

$$\bullet 1 + 2i = z$$

$$\sin \theta = \frac{2}{\sqrt{5}}$$

$$\cos \theta = \frac{1}{\sqrt{5}}$$

$$\theta = \arcsin \frac{2}{\sqrt{5}}$$

$$\theta = \frac{\pi}{6}$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$r = 1$$

$$\sin \frac{\pi}{6} = \frac{1}{2}$$

$$z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$r = 2$$

$$z = \sqrt{3} + i$$

$$z = r (\cos \theta + i \sin \theta)$$

$$z_1 z_2 = \text{compute at home}$$