The argument of a complex number is only determined up to multiples of 2π.

$$\arg(z) = \theta + 2n\pi$$

$$\mathrm{Arg}(z) = \theta \in \left]-\pi, \pi\right]$$

principal argument.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2\big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\big)$$

$$= r_1 r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$$

$$|z_1 z_1| = |z_1|\cdot|z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

<u>Application of the above and the triangle inequality</u>:

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

Show: when $|z| \longrightarrow \infty$

$|P(z)| \longrightarrow \infty$ also.

More precisely, if $|z|$ is big, then $|P(z)|$ is big.

$$P(z) = z^n\left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)$$

$$|P(z)| = |z^n|\left|a_n + \cdots + \frac{a_0}{z^n}\right|$$

$$= |z|^n\left|a_n + \cdots + \frac{a_0}{z^n}\right|$$

$$a_n + \underbrace{\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}}_{w.} = a_n + w.$$

$$|a_n + w| \geqslant \big||a_n| - |-w|\big| = \big||a_n| - |w|\big|$$

If $|z| > R$ large real number.

then $\dfrac{1}{|z|} < \dfrac{1}{R}.$

take R large enough so that

$$\frac{|a_{n-1}|}{|z|} = \left|\frac{a_{n-1}}{z}\right| \leq \left|\frac{a_n}{2n}\right|$$

$$\left|\frac{a_{n-2}}{z^2}\right| \leq \left|\frac{a_n}{2n}\right|$$

$$\vdots$$

$$\left|\frac{a_0}{z^n}\right| \leq \left|\frac{a_n}{2n}\right|$$

$$|w| = \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \leq \left|\frac{a_{n-1}}{z}\right| + \cdots + \left|\frac{a_0}{z^n}\right|$$

$$\leq \frac{|a_n|}{2n} + \cdots + \frac{|a_n|}{2n} = \frac{|a_n|}{2}$$

So $|a_n + w| \geq \big||a_n| - |w|\big|$

$$\geq |a_n| - |w| \geq |a_n| - \frac{|a_n|}{2}$$

$$\geq \frac{|a_n|}{2}.$$

$$|P(z)| = |z|^n \, |a_n + w| \geq |z|^n \frac{|a_n|}{2}$$

$$|z| \geq R \Rightarrow |P(z)| > R^n \frac{|a_n|}{2}$$

So $\lim_{|z| \to \infty} |P(z)| = \infty$.

<u>Back to polar forms</u>:

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2))$$

$$z^2 = r^2 (\cos(2\theta) + i \sin(2\theta))$$

$$\vdots$$

$$z^n = r^n (\cos(n\theta) + i \sin(n\theta))$$

<u>De Moivre's formula</u>:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

<u>Euler's formula</u>:

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

$$= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta .$$

$$f(\theta) = e^{i\theta} \qquad f'(\theta) = i e^{i\theta}$$

$$f''(\theta) = i^2 e^{i\theta} = -e^{i\theta} = -f(\theta)$$

$$f(0) = 1 \qquad f'(0) = i$$

$f'' = -f$ $\quad \left\{ \begin{array}{l} \cos\theta \\ \sin\theta \end{array} \right.$ ~~sat~~ satisfies the equation

$$f(\theta) = A\cos\theta + B\sin\theta$$

$$1 = f(0) = A \qquad i = f'(0) = B$$

$$f'(\theta) = -A\sin\theta + B\cos\theta$$

$$\Rightarrow f(\theta) = \cos\theta + i\sin\theta .$$

$$\boxed{z = re^{i\theta}} \qquad |z| = r \qquad |e^{i\theta}| = 1.$$

$z \in$ circle of center z_0 and radius R:

$$\iff \quad |z - z_0| = R$$

$$\iff \quad z - z_0 = Re^{i\theta}$$

$$\iff \quad z = z_0 + Re^{i\theta}$$

parametrization of the circle: the parameter is θ.

Roots: $\quad z = re^{i\theta}.$

$$z^2 = r^2 e^{2i\theta}$$

$$\vdots$$

$$z^n = r^n e^{ni\theta}$$

If w is an n-th root of z, then $|w| = \sqrt[n]{|z|}$ and

$$\arg(w) = \frac{1}{n}\arg(z):$$

$$w^n = z. \qquad w = se^{i\alpha}$$

$$w^n = z \qquad s^n e^{in\alpha} = r e^{i\theta}$$

$$\Rightarrow \quad s^n = r \quad \Rightarrow \quad s = \sqrt[n]{r}$$

$$n\alpha = \theta + 2k\pi$$

$$\alpha = \frac{\theta}{n} + \frac{1}{n} 2k\pi$$

$z = 1 + i \qquad \sqrt{1+i} \;?$

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}.$$

$$z = \sqrt{2} \, \frac{(1+i)}{\sqrt{2}} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$\cos\theta = \frac{1}{\sqrt{2}} \qquad \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4} + 2k\pi \qquad z = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$e^{i\frac{\pi}{4} + i2k\pi} = e^{i\pi/4} \, e^{i2k\pi}$$

$$e^{i2k\pi} = \cos(2k\pi) + i\sin(2k\pi)$$
$$= 1$$

$$w^2 = z \qquad |w| = \sqrt{|z|} = \sqrt[4]{2}.$$

$$\arg(w) = \alpha \qquad 2\arg w = \arg(z)$$

$$2\alpha = \theta + 2k\pi$$

$$\alpha = \frac{\theta}{2} + k\pi = \frac{\pi}{8} + k\pi.$$

$$w = \sqrt[4]{2}\, e^{i\frac{\pi}{8}} \quad \text{or} \quad w = \sqrt[4]{2}\, e^{i\left(\frac{\pi}{8}+\pi\right)}$$

$$= \sqrt[4]{2}\, e^{i\pi/8} e^{i\pi}$$

$$e^{i\pi} = \cos\pi + i\sin\pi \qquad = -\sqrt[4]{2}\, e^{i\pi/8}$$

$$= -1$$

principal argument for

$$\frac{\pi}{8} + \pi - 2\pi = -\frac{7\pi}{8}$$

$$w = \sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

$$\frac{\pi}{8} = \frac{1}{2}\left(\frac{\pi}{4}\right)$$

half-angle formulas:

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} \qquad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+1/\sqrt{2}}{2}} \qquad \sin\frac{\pi}{8} = \sqrt{\frac{1-1/\sqrt{2}}{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2} \qquad = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$w = \sqrt[4]{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i\,\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

or $$w = -\sqrt[4]{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i\,\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

$\sqrt{i}$? $\qquad |i| = 1 \qquad i = e^{i\theta}$

$$\cos\theta = 0 \qquad \sin\theta = 1 \qquad \theta = \frac{\pi}{2}$$

$$i = e^{i\pi/2} \qquad \sqrt{i} = e^{i\alpha}$$

$$2\alpha = \frac{\pi}{2} + 2k\pi$$

$$\alpha = \frac{\pi}{4} + k\pi \qquad \begin{cases} e^{i\pi/4} \\ e^{i(\pi/4+\pi)} \end{cases}$$

$$\begin{cases} e^{i\pi/4} \\ -e^{i\pi/4} \end{cases}$$

$\sqrt[3]{i}$? $\qquad w^3 = i \qquad w = e^{i\alpha}$

$$\left(e^{i\alpha}\right)^3 = e^{i\pi/2}$$

$$3\alpha = \frac{\pi}{2} + 2k\pi$$

$k=0 \quad \alpha = \frac{\pi}{6} \qquad k=1 \quad \alpha = \frac{\pi}{6} + \frac{2\pi}{3}$

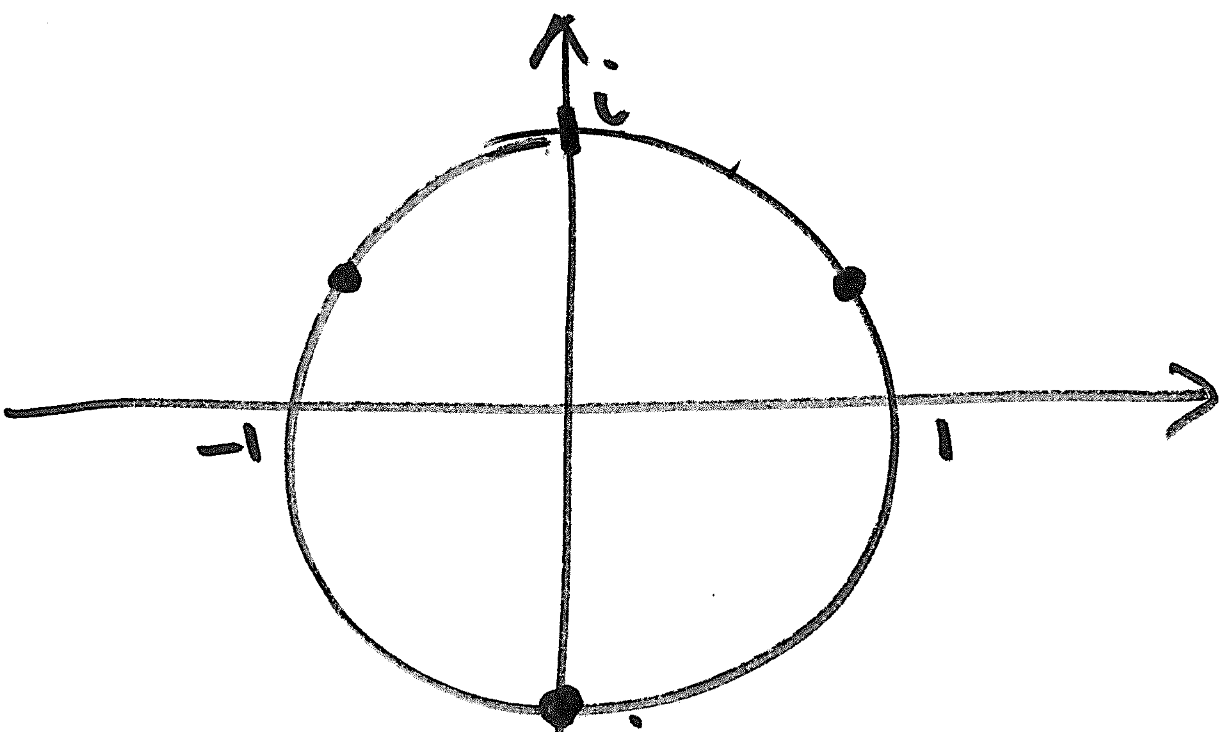
$k=2 \quad \alpha = \frac{\pi}{6} + \frac{4\pi}{3} \qquad k=3$

$\alpha = \frac{\pi}{6} + 2\pi$

$$n\alpha = \theta + 2k\pi$$

$$\alpha = \frac{\theta}{n} + \frac{k}{n} 2\pi$$

$\sqrt[3]{i} \quad , \quad \alpha = \frac{\pi}{6} \text{ or } \alpha = \frac{5\pi}{6}, \alpha = \frac{3\pi}{2}$



$\alpha = \frac{\theta}{n} + \frac{k}{n} 2\pi \qquad \cancel{k=0}$

$k=0 \quad \alpha = \frac{\theta}{n} \longleftarrow$ principal root

$k=1 \quad \alpha = \frac{\theta}{n} + \frac{2\pi}{n}$

$k = 2 \qquad \alpha = \frac{\theta}{n} + 2 \cdot \frac{2\pi}{n}$

$k = 3 \qquad \alpha = \frac{\theta}{n} + 3 \cdot \frac{2\pi}{n}$

The n-th roots are evenly distributed around the circle: the angle between any two of them is $\frac{2\pi}{n}$.