Regions in complex plane:

<u>in $\mathbb{R}$</u> : $a \in \mathbb{R}$ $\quad]a-\varepsilon, a+\varepsilon[$

ε-neighborhood

n punctured ε-neighborhood:

$$]a-\varepsilon, a+\varepsilon[\setminus \{a\}$$

$$=]a-\varepsilon, a[\cup]a, a+\varepsilon[$$

~~in ℂ~~ n: $\quad \{x : |x-a| < \varepsilon\}$.

$$\{x : 0 < |x-a| < \varepsilon\}$$

<u>in $\mathbb{C}$</u>: <u>$a \in \mathbb{C}$</u> $\{z : |z-a| < \varepsilon\}$ open disc

$\{z : 0 < |z-a| < \varepsilon\}$ punctured

$S \subset \mathbb{C}$ $\qquad S$ subset.

<u>**Definition**</u>: S is open if for every $z \in S$, $\exists\, \varepsilon$ s.t. S contains an ε-neighborhood of z

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$S \subset \mathbb{C}$, S arbitrary

<u>**Def.**</u>: $a \in S$ $\quad a$ is an interior point if $\exists\, \varepsilon > 0$ s.t. S contains the ε-neighborhood of a.

$a \in S$

Def: a is an exterior point of S if there is an ε-neighborhood of a which does not contain any point of S. ($\Rightarrow a \notin S$)

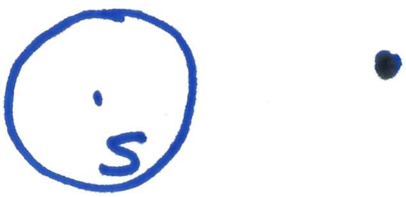
Def: a is a boundary point of S if it is neither interior nor exterior.
In other words, every ε-neighborhood of a contains points that are in S and points that are not in S.

Def: The boundary of S, denoted ∂S is the set of all the boundary points of S.

Def: We say S is closed if it contains its boundary. e.g.: $\{z : |z-a| \leq \varepsilon\}$

Def: The closure of S is

$$\boxed{S \cup \partial S}$$



__Note:__ S open $\iff$ every point of S is interior.

__Note:__ S open $\iff$ the complement of S is closed

The complement of S is $\mathbb{C}\setminus S$: the set of points __not__ in S.

$$S^c := \mathbb{C}\setminus S,$$

$z \in S$, $\quad S \supset \varepsilon$-neighborhood of z

$\Updownarrow \qquad\qquad \|$

$z \notin S^c \qquad (S^c)^c \supset \varepsilon$-neighborhood of z

$\qquad \|$

$z \notin T \qquad T^c \supset \varepsilon$-neighborhood of z

$z \in S$

so z is exterior to $T = S^c$
$\iff$ z is interior to S.

S open $\iff$ S does not contain any point of $\partial S = \partial T$

$\iff$ T contains $\partial S = \partial T$

$\iff$ T closed.

Def: S is bounded if $\exists\, R > 0$ s.t. $S \subset$ disc of radius R.

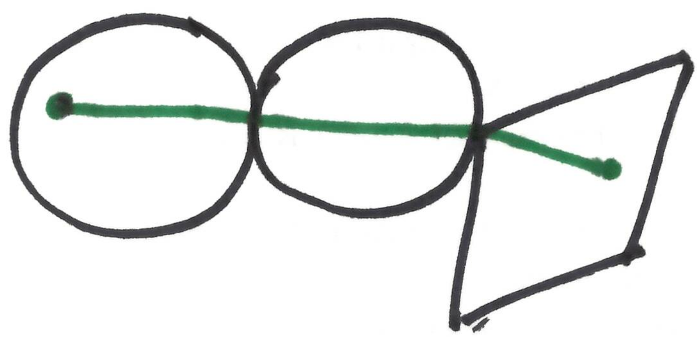
S is compact if it is closed and bounded.

S is path-connected if any two points of S can be joined by a {polygonal / or continuous} path inside S.

Examples: Upper half planes:

open: $\operatorname{Im} z > 0$.

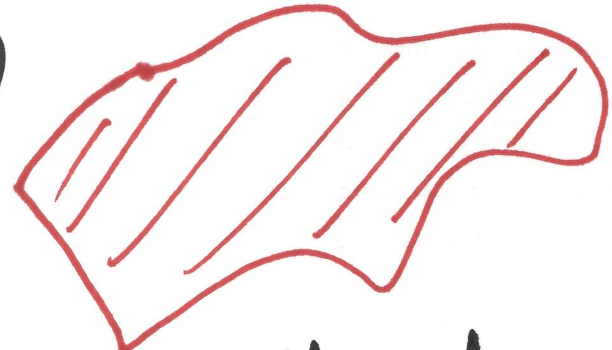
closed: $\operatorname{Im} z \geq 0$ closed, unbounded

e.g.:  path-connected.

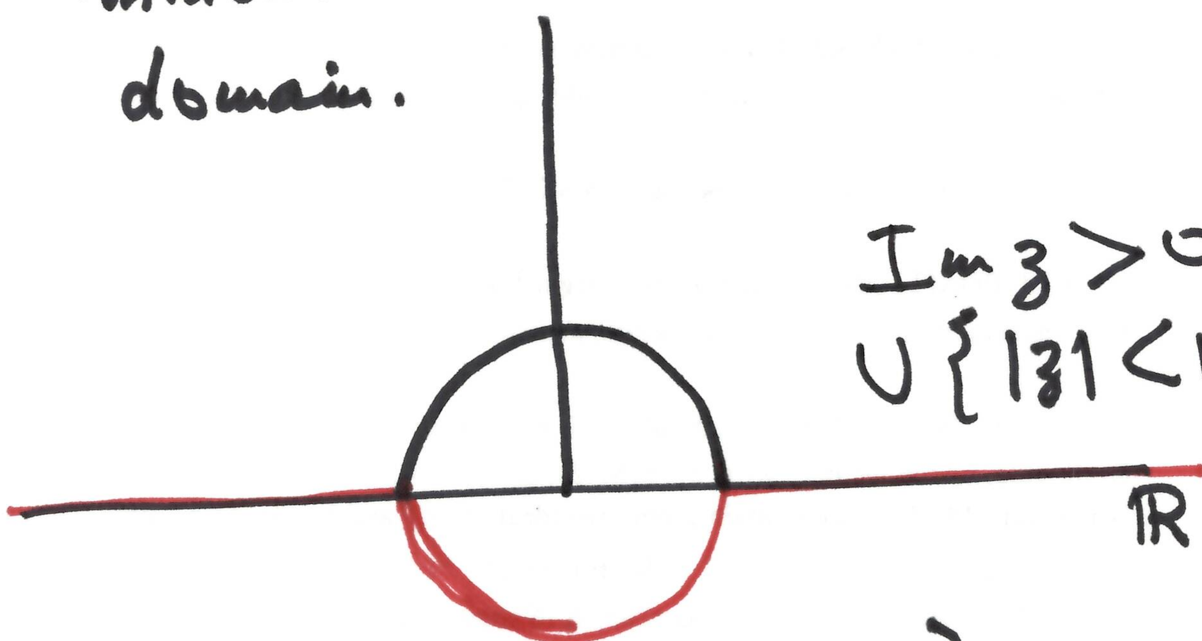
S open: S is ~~connect~~ connected if it is NOT a disjoint union of two other open sets.

Def: A domain is a connected open subset of $\mathbb{C}$

Def: A region is a domain together with some boundary points.

e.g.: ① 

without the boundary, it is a domain.

② 

$\operatorname{Im} z > 0$
$\cup \{|z|<1\}$.

③ $S = \{x+iy \mid x, y \in \mathbb{Q}\}$.

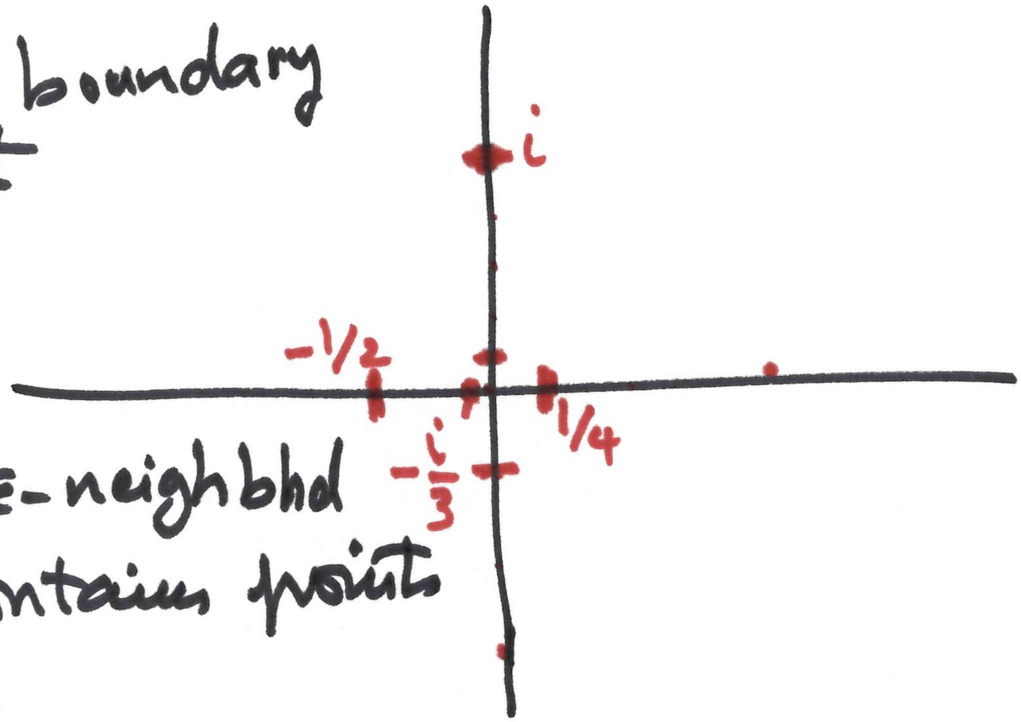
does not have any interior or exterior points.

Neither closed nor open: $\sqrt{2}$ is a boundary point but not in S: S not closed.

1 is a boundary point of S^c, but not in S^c: so S^c is not closed: S not open

④ $S = \left\{ \frac{i^n}{n} : n = 1, 2, 3, 4, \dots \right\}$. $\not\ni 0$

0 is a boundary point



every ε-neighbhd of 0 contains points of S.

All points of S are isolated:

$a \in S$ is isolated if $\exists$ ε-neighbhd of a which intersects S only in a

0 is an accumulation point of S.

<u>Def</u>: An accumulation point for S is $a \in \mathbb{C}$ s.t. every punctured ε-neighborhood of a contains points of S.

Ex: S is closed $\iff$ S contains all of its accumulation points.

⑤ $\mathrm{Im}\left(\frac{1}{z}\right) > 1$ (book)

$\iff \left|z + \frac{i}{2}\right| < \frac{1}{2}$

Extreme value theorem:

A continuous function $f : D \to \mathbb{R}$ where D is compact has a maximum and a minimum on D.

$c \in \mathbb{C}$ $\quad z_0 = 0 \quad z_1 = z_0^2 + c$

$\ldots \quad z_{n+1} = z_n^2 + c \quad \ldots$

$\mathcal{M}$ = Mandelbrot set $= \{c \mid |z_n| \not\to \infty\}$.

can prove $|z_n| \not\to \infty \iff |z_n| \leq 2 \ \forall n$

$\mathcal{M} \ni 0, -1$, any z within distance $\frac{1}{4}$ of 0 or -1.

Functions: $S \subset \mathbb{C}$

a function is a map $f : S \to \mathbb{C}$

If we do not specify the domain of definition of f, then the domain is the largest set where f makes sense.

e.g.: $f(z) = z^2 + 1$ domain $= \mathbb{C}$

$f(z) = \frac{1}{z}$ domain $= \mathbb{C} \setminus \{0\}$

$z = x + iy$ $\quad f(z) = u + iv$

$$f(x,y) = u(x,y) + i v(x,y).$$

e.g.: $f(z) = z^2$ $\quad z = x + iy$

$$z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy).$$

$$f(z) = (x^2 - y^2) + i(2xy).$$