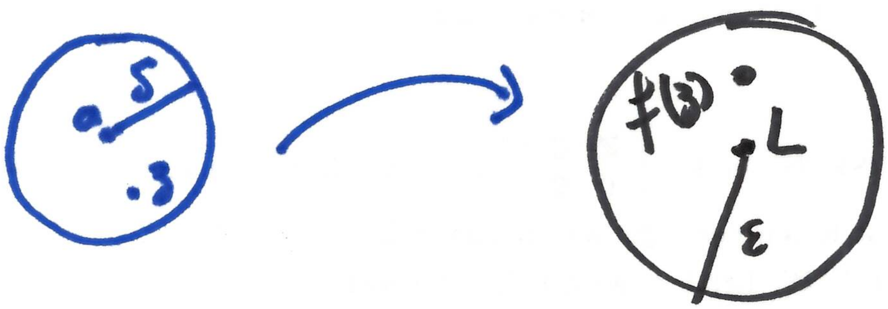
Limits: If the domain of f contains points arbitrarily close to $a \in \mathbb{C}$, we can talk about $\lim_{z \to a} f(z) = L$

This means: $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$
s.t. $|z-a| < \delta \Rightarrow |f(z) - L| < \varepsilon$



e.g.: $f(z) = z^2$ $\quad \lim_{z \to 1} z^2 = 1$

$\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t.
$|z-1| < \delta \Rightarrow |z^2 - 1| < \varepsilon$

$$|z^2 - 1| = |(z-1)(z+1)| = |z-1| \cdot |z+1|$$

first take z within radius 1 of 1:
take: $|z-1| < 1$

$$|z+1| = |z-1+2| \leq |z-1|+2 < \delta+2$$

take $\delta \leq 1$ then ≤ 3

take $\delta = \operatorname{Min}\left\{\frac{\varepsilon}{3}, 1\right\}$

then $|z-1| < \delta \Rightarrow$

$$|z^2-1| = |z-1|\,|z+1| < \delta \cdot 3 \leq \frac{\varepsilon}{3} \cdot 3$$

$$\Rightarrow |z^2-1| < \varepsilon$$

Uniqueness of limits:

If $\lim\limits_{z \to a} f(z) = w_1$ & $\lim\limits_{z \to a} f(z) = w_2$

then $w_1 = w_2$.

Proof: $\forall\, \varepsilon > 0$, $\exists\, \delta_1$ s.t.

$$|z-a| < \delta_1 \Rightarrow |f(z) - w_1| < \varepsilon$$

$\forall\, \varepsilon > 0$, $\exists\, \delta_2$ s.t.

$$|z-a| < \delta_2 \Rightarrow |f(z) - w_2| < \varepsilon$$

$$|w_1 - w_2| = |w_1 - f(z) + f(z) - w_2|$$

$\leq |f(z)-w_1| + |f(z)-w_2|$

$\forall\ \varepsilon > 0 \quad \exists\ \delta_1$ s.t. $|z-a| < \delta_1$

$\Rightarrow |f(z)-w_1| < \varepsilon$

$\exists\ \delta_2$ s.t. $|z-a| < \delta_2$

$\Rightarrow |f(z)-w_2| < \varepsilon$

take $|z-a| < \text{Min}\{\delta_1, \delta_2\}$

$\Rightarrow |f(z)-w_1| < \varepsilon$ & $|f(z)-w_2| < \varepsilon$

$\Rightarrow |w_1-w_2| < 2\varepsilon$

$\forall\ \varepsilon > 0 \quad |w_1-w_2| < 2\varepsilon$

$\Rightarrow |w_1-w_2| = 0 \qquad \square$

<u>Limit laws:</u> <u>Book:</u>

$f(z) = f(x,y) = u(x,y) + i\,v(x,y)$

$z = x+iy$

$\lim\limits_{z\to a} f(z) = L \Leftrightarrow \lim\limits_{z\to a} u = \operatorname{Re} L$

& $\lim\limits_{z\to a} v = \operatorname{Im} L$

e.g. $f(z) = z^2 = x^2 - y^2 + i(2xy)$

$\lim_{z \to 1} (x^2 - y^2) = 1, \quad \lim_{z \to 1} 2xy = 0$

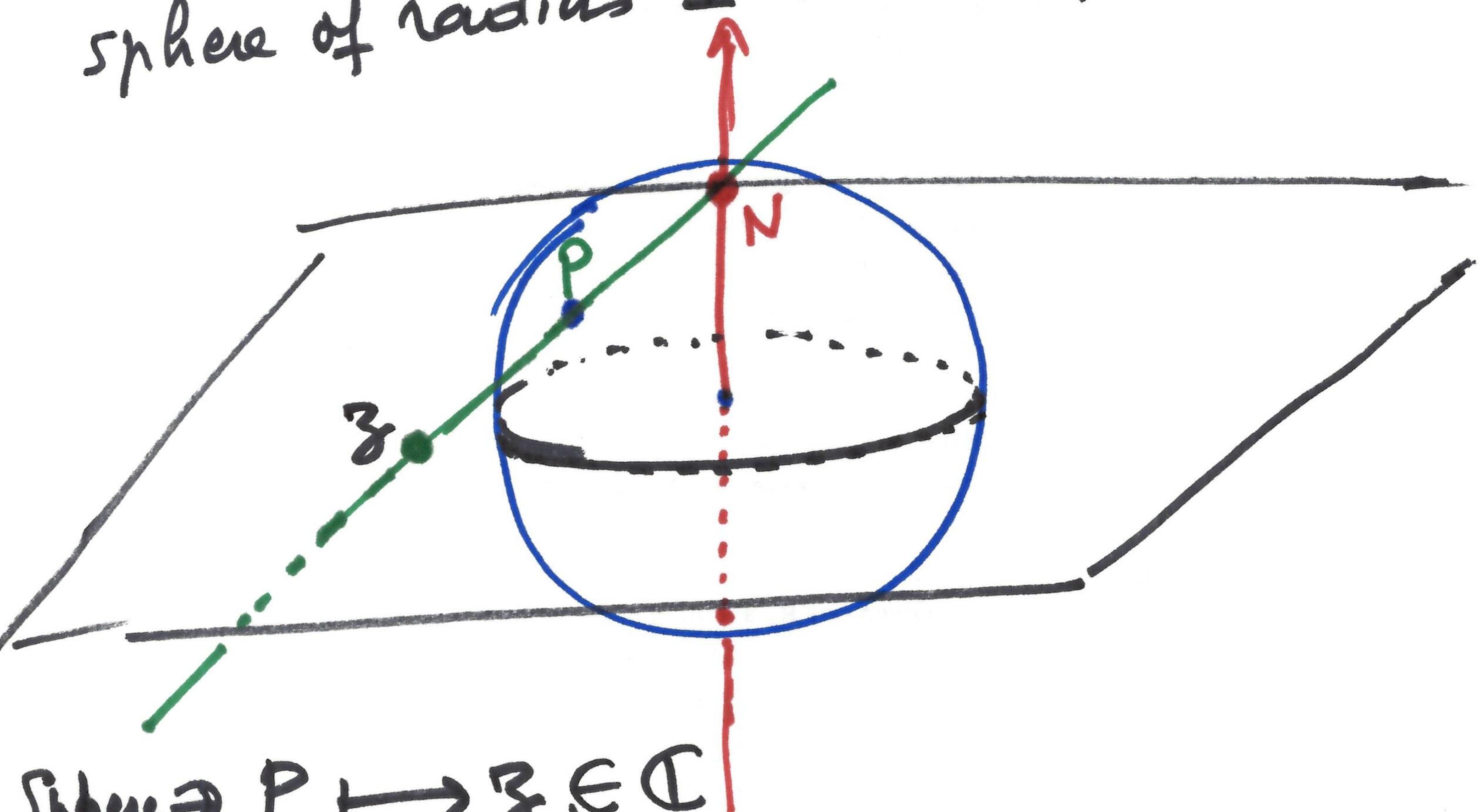
$z = x + iy \longrightarrow 1$ means $x \to 1, y \to 0$

$\Rightarrow \lim_{z \to 1} z^2 = 1$

$z \longrightarrow \infty$ means $|z| \longrightarrow \infty$

<u>Visualize:</u> <u>Stereographic projection:</u>

sphere of radius $\underline{1}$: unit sphere.



Sphere $\ni P \longmapsto z \in \mathbb{C}$

$P \longleftarrow\!\shortmid\ z$

coordinates of P in terms of z:

$$\left(\frac{2\,\mathrm{Re}\,z}{1+|z|^2}, \frac{2\,\mathrm{Im}\,z}{1+|z|^2}, \frac{-1+|z|^2}{1+|z|^2}\right)$$

$f(z)$. $\lim_{z\to a} f(z) = \infty$.

<u>means</u>: $\forall\ R>0$, $\exists\ \delta>0$

s.t. $|z-a|<\delta \Rightarrow |f(z)|>R$.

<u>another way</u>: $\lim_{z\to a}\frac{1}{f(z)} = 0$

$\forall\ \varepsilon>0$, $\exists\ \delta>0$ s.t.

$|z-a|<\delta \Rightarrow \left|\frac{1}{f(z)}\right| < \varepsilon$.

$\lim_{z\to\infty} f(z) = L$

$\forall\ \varepsilon>0$, $\exists\ R>0$ s.t.

$|z|>R \Rightarrow |f(z)-L|<\varepsilon$

$z\to\infty$ when $\frac{1}{z}\to 0$

$\lim_{z\to\infty} f(z) = L$:

$\forall \ \varepsilon > 0 \ , \ \exists \ \delta > 0$ s.t.

$\left|\frac{1}{z}\right| < \delta \Rightarrow |f(z) - L| < \varepsilon$

$\lim_{z\to\infty} f(z) = \infty.$

$\forall \ R > 0, \ \exists \ T > 0$ s.t.

$|z| > T \Rightarrow |f(z)| > R$

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e.g. 1 $\lim_{z\to 2} \frac{1}{z-2} = \infty.$

$\forall \ R > 0, \ \exists \ \delta > 0$ s.t.

$|z-2| < \delta \Rightarrow \frac{1}{|z-2|} > R.$

Choose $\delta \leqslant \frac{1}{R}$, then $|z-2| < \delta$

$\Rightarrow |z-2| < \frac{1}{R} \Rightarrow \frac{1}{|z-2|} > R$

$$\lim_{z\to\infty} \frac{z^2+1}{3z^2-1} = \frac{1}{3}$$

$\forall \varepsilon > 0$, $\exists R > 0$ s.t.

$$|z| > R \Rightarrow \left| \frac{z^2+1}{3z^2-1} - \frac{1}{3} \right| < \varepsilon$$

simplify: $\left| \frac{3z^2+3-3z^2+1}{3(3z^2-1)} \right|$

$$\frac{4}{3|3z^2-1|} \overset{?}{<} \varepsilon$$

③ $\boxed{\frac{4}{3\varepsilon} < |3z^2-1|}$

$|z| > R \quad (z^2) > R^2 \quad |3z^2| > 3R^2$

$\cancel{|3z^2|-1 > 3R^2-1}$

①

~~Need~~ $\cancel{3R^2-1}$

$\boxed{|3z^2-1| \geq \big||3z^2|-1\big| > 3R^2-1}$

Need $\boxed{3R^2-1 > \frac{4}{3\varepsilon}.}$ ②

① + ② $\rightsquigarrow$ ③

$$\boxed{3R^2 > 1 + \frac{4}{3\varepsilon}}$$

$$R^2 > \frac{1}{3} + \frac{4}{9\varepsilon}$$

If $\boxed{R > \sqrt{\frac{1}{3}+\frac{4}{9\varepsilon}}}$, then

$3R^2-1 > \frac{4}{3\varepsilon}$, then

$|3z^2-1| > 3R^2-1 > \frac{4}{3\varepsilon}$, then

$$\left|\frac{z^2+1}{3z^2-1} - \frac{1}{3}\right| < \varepsilon.$$

$\square$

$f : S \longrightarrow \mathbb{C}$ $\quad$ S region in $\mathbb{C}$

We say f is continuous at $a \in S$ if $\lim_{z \to a} f(z) = f(a)$

$\lim_{z \to a} f(z)$ can exist even if f is not defined at a.

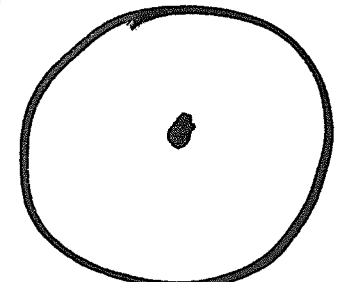
but we can only talk about f being continuous at a when f is well defined at a.

Extreme value theorem:

f continuous on a compact set $K \subset \mathbb{C}$ then $|f| : K \to \mathbb{R}$ is continuous and has a maximum and a minimum on K.

In particular, f is bounded on K, meaning $\exists M > 0$ s.t. $|f(z)| \leq M$ $\forall z \in K$. (f cannot have ∞ limits on K)

Differentiability:



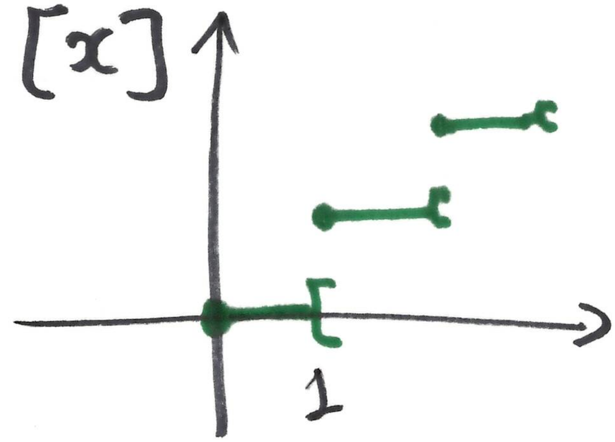
unit disc

z^2

$|z^2| \leq 1$

One way of proving $\lim_{z\to a} f(z)$ does not exist is to show one obtains different limits by approaching a in different ways.

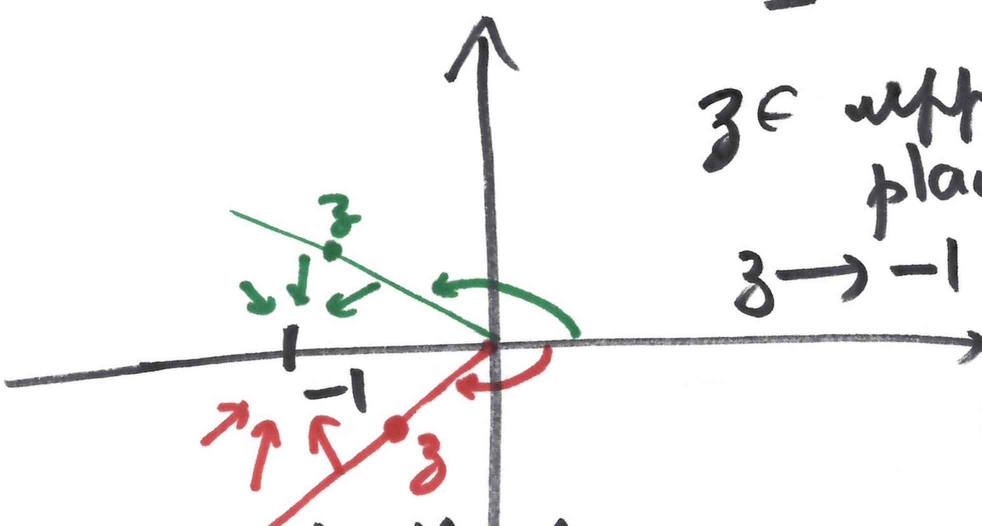
$f(x) = [x]$



$f(z) = [|z|]$

<u>e.g.</u>: $f(z) = \operatorname{Arg} z$, $\quad \operatorname{Arg}(-1) = \pi$

$\in]-\pi, \pi]$



$z \in$ upper half plane

$\lim_{z \to -1} \operatorname{Arg} z = \pi$

$z \in$ lower half plane $z \to -1$

$\lim \operatorname{Arg} z = -\pi$.

Differentiability: We say $f : S \to \mathbb{C}$ is differentiable at $z_0 \in \mathrm{Int}(S)$ ~~iff~~ with derivative $f'(z_0)$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$