$f : S \longrightarrow \mathbb{C}$

$z_0 \in$ Interior of S

<u>Def</u>: The derivative of f at z_0, if it exists is $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} =: f'(z_0)$

If $f'(z_0)$ exists, we say f is differentiable at z_0.

<u>e.g.</u>: $f(z) = z$. $f(z) = z^n$ $n \geq 0$
differentiable.

$f(z) = \bar{z}$ not differentiable anywhere (book).

$f(z) = |z|^2 = z\bar{z}$ $S = \mathbb{C}$

$$\lim_{\Delta z \to 0} \underbrace{\left(\frac{f(z + \Delta z) - f(z)}{\Delta z} \right)}_{\|} ?$$

$$\neq \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \frac{z\bar{z} + z\overline{\Delta z} + (\Delta z)\bar{z} + \Delta z\overline{\Delta z} - z\bar{z}}{\Delta z}$$

$$= \left(z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} \right)$$

$$\lim_{\Delta z \to 0} \left(\qquad \right) = \lim_{\Delta z \to 0} \left(\bar{z} + \frac{\overline{\Delta z}}{\Delta z} z \right)$$

e.g. $\Delta z \in \mathbb{R}$ $\quad \lim_{\Delta z \to 0} (\bar{z} + z) = 2\,\mathrm{Re}\,z.$

$\Delta z \in i\mathbb{R}$ $\quad \lim_{\Delta z \to 0} (\bar{z} - z) = -2\,\mathrm{Im}\,z.$

more generally: $\quad \Delta z = r e^{i\theta}$

$$\overline{\Delta z} = r e^{-i\theta}$$

$$\frac{\overline{\Delta z}}{\Delta z} = \frac{r e^{-i\theta}}{r e^{i\theta}} = e^{-2i\theta}$$

$\lim_{r \to 0} \left(\bar{z} + e^{-2i\theta} z \right)$ undefined i θ is not fixed

unless $z = 0$.

<u>So:</u> $|z|^2$ is differentiable at $z = 0$ and nowhere else.

e.g. $f(z) = z^n \quad n \geq 0 \quad S = \mathbb{C}$

$f'(z) = n z^{n-1}$ f differentiable everywhere

$n < 0 \quad S = \mathbb{C} \setminus \{0\}$. similar.

Usual rules apply:

e.g. $(f \pm g)' = f' \pm g'$

$(fg)' = f'g + fg'$

Chain rule: (nice proof in book)

$$\boxed{(g \circ f)'(z) = g'(f(z)) \cdot f'(z).}$$

or $f(z) = w \quad g(w) = W$

$$\boxed{\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz}}$$

$\boxed{w = f(z)} \quad w_0 = f(z_0)$

if $f'(z_0)$ exists, then: $w - w_0 \sim f'(z_0)(z - z_0)$

$f'(z_0) = \lim_{z \to z_0} \frac{w - w_0}{z - z_0}$ when $z \sim z_0$

geometrically: $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$\boxed{z_1} \cdot \boxed{z_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

So: multiplication by $f'(z_0)$:

dilation by $|f'(z_0)|$ +

rotation by $\arg f'(z_0)$.

Cauchy-Riemann equations:

D'Alembert (1752), Euler (1797)

Cauchy (1814), Riemann (1851)

$f(x+iy) = u(x+iy) + i v(x+iy)$

or $f(x,y) = u(x,y) + i v(x,y)$.

Theorem: If $f'(z_0)$ exists at $z_0 = x_0 + iy_0$

then u & v are differentiable with respect to x & y at x_0 & y_0 (meaning the partial derivatives exist)

and $u_x = v_y$ & $u_y = -v_x$

& $f'(z_0) = u_x + i v_x$ at (x_0, y_0)

~~Proof: f(z)~~ $u_x = \frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y} = u_y$, ...

Proof: $f(z) - f(z_0) =$

$$u(x,y) + i v(x,y) - u(x_0,y_0) - i v(x_0,y_0)$$

$$\lim_{z \to z_0} \frac{u(x,y) - u(x_0,y_0) + i\left(v(x,y) - v(x_0,y_0)\right)}{x - x_0 + i(y - y_0)}$$

f differentiable means the limit exists.

We can approach z_0 any way:

fix $y = y_0$

$$\lim_{x \to x_0} \left[\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + \frac{i\left(v(x,y_0) - v(x_0,y_0)\right)}{x - x_0} \right]$$

exists.

$$\Rightarrow \lim_{x \to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} \text{ exists}$$

$$= u_x(x_0,y_0)$$

$$\& \lim_{x \to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0} \text{ exists}$$

$$= v_x(x_0, y_0)$$

Similarly, fix $x = x_0$ to obtain $u_y(x_0, y_0)$ exists & $v_y(x_0, y_0)$ exists.

When we fixed $y = y_0$, we got

$$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$$

when we fix $x = x_0$, we get:

$$f'(z_0) = \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0) + i\big(v(x_0, y) - v(x_0, y_0)\big)}{i(y - y_0)}$$

$$= \lim_{y \to y_0} \frac{-i\big(u(x_0, y) - u(x_0, y_0)\big) + v(x_0, y) - v(x_0, y_0)}{y - y_0}.$$

$$= -i\, u_y(x_0, y_0) + v_y(x_0, y_0)$$

<u>So</u>: $u_x(x_0, y_0) = v_y(x_0, y_0)$

& $u_y(x_0, y_0) = -v_x(x_0, y_0)$ $\square$

<u>e.g.</u>: $f(z) = z^2 = x^2 - y^2 + i(2xy)$

$z = x + iy$

$u = x^2 - y^2 \qquad v = 2xy.$

$u_x = 2x \qquad v_x = 2y.$

$f'(z) = 2z = 2x + i(2y) = u_x + iv_x$

$v_y = 2x = u_x, \quad u_y = -2y = -v_x$

~~Theorem~~: In Calculus, you called a vector valued function of several variables differentiable when the partial derivatives existed and were continuous on a neighborhood. This ensures that the function is approximated by the linear term of its Taylor expansion.

Similarly, we need a little more than the existence of $f'(z_0)$ for the converse of C.R.:

Theorem: $f(z) = f(x,y) = u(x,y) + iv(x,y)$

Suppose u, v are well-defined on

some ε-neighborhood of $z_0 = x_0 + iy_0$ and their partial derivatives exist on that ε-neighborhood and are continuous at z_0. Also suppose that

$$u_x(x_0,y_0) = v_y(x_0,y_0) \ \& \ u_y(x_0,y_0) = -v_x(x_0,y_0)$$

then f is differentiable at z_0 with $f'(z_0) = u_x(z_0) + i\, v_x(z_0)$

<u>Proof</u>: near z_0 f:

$$\cancel{u(x,y) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)}$$

$$\cancel{+ \varepsilon_1 (x - x_0)}$$

~~where~~ $\cancel{\varepsilon_1 \to 0}$ ~~as~~ $\cancel{x \to x_0}$.

$$\begin{aligned} u(x,y) = {} & u(x_0,y_0) \\ & + u_x(x_0,y_0)(x-x_0) \\ & + u_y(x_0,y_0)(y-y_0) \\ & + \cancel{\varepsilon_1 (x-x_0) + \varepsilon_2 (y-y_0)} \\ & \varepsilon_1 \sqrt{(x-x_0)^2 + (y-y_0)^2} \end{aligned}$$

$$
\begin{aligned}
v(x,y) = {} & v(x_0,y_0) \\
& + v_x(x_0,y_0)(x-x_0) \\
& + v_y(x_0,y_0)(y-y_0) \\
& + \varepsilon_2 \underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}}_{\varepsilon_2|\Delta z|}
\end{aligned}
$$

where $\lim_{z \to z_0} \varepsilon_1 = \lim_{z \to z_0} \varepsilon_2 = 0$

$$
\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}
$$

$$
= \lim_{z \to z_0} \frac{u(x,y)-u(x_0,y_0)+i\left(v(x,y)-v(x_0,y_0)\right)}{x-x_0+i(y-y_0)}
$$

$$
=
$$