$$v(x,y) = v(x_0, y_0) + v_x(x_0, y_0)(x - x_0) + v_y(x_0, y_0)(y - y_0) + \underbrace{\varepsilon_2 \sqrt{(x-x_0)^2 + (y-y_0)^2}}_{\varepsilon_2 |\Delta z|}$$

where $\lim\limits_{z \to z_0} \varepsilon_1 = \lim\limits_{z \to z_0} \varepsilon_2 = 0$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{u(x,y) - u(x_0, y_0) + i\left(v(x,y) - v(x_0, y_0)\right)}{x - x_0 + i(y - y_0)}$$

$$= \lim_{z \to z_0} \left[\frac{u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0) + \varepsilon_1 \sqrt{(x-x_0)^2 + (y-y_0)^2}}{x - x_0 + i(y - y_0)} + \frac{i\left(v_x(x_0, y_0)(x - x_0) + v_y(x_0, y_0)(y - y_0) + \varepsilon_2 \sqrt{(x-x_0)^2 + (y-y_0)^2}\right)}{x - x_0 + i(y - y_0)} \right]$$

$$= \lim_{z \to z_0} \left[\frac{u_x(x_0,y_0)\left[(x-x_0)+i(y-y_0)\right]}{z-z_0} + \frac{i v_x(x_0,y_0)\left[(x-x_0)+i(y-y_0)\right]}{z-z_0} + \frac{(\varepsilon_1+\varepsilon_2)\sqrt{(x-x_0)^2+(y-y_0)^2}}{z-z_0} \right.$$

$$= u_x(x_0,y_0) + i v_x(x_0,y_0) + \lim_{z \to z_0} \frac{(\varepsilon_1+\varepsilon_2)|z-z_0|}{z-z_0}$$

$$\left| \frac{(\varepsilon_1+\varepsilon_2)|z-z_0|}{z-z_0} \right| = |\varepsilon_1+\varepsilon_2| \cdot \frac{|z-z_0|}{|z-z_0|}$$

$$= |\varepsilon_1+\varepsilon_2|$$

$$\lim_{z \to z_0} \left| \frac{(\varepsilon_1+\varepsilon_2)|z-z_0|}{z-z_0} \right| = \lim_{z \to z_0} |\varepsilon_1+\varepsilon_2| = 0$$

So $\displaystyle \lim_{z \to z_0} \frac{(\varepsilon_1+\varepsilon_2)|z-z_0|}{z-z_0} = 0$

So $\displaystyle \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = u_x(z_0) + i v_x(z_0)$

$\square$

e.g.: $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(z)=e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

$$= e^x\cos y + i e^x \sin y$$

the components ($\mathrm{Re} f(z)$ and $\mathrm{Im} f(z)$) are differentiable with continuous partials. (in fact differentiable infinitely many times)

$$u_x = e^x\cos y \quad = \quad v_y = e^x\cos y$$

$$u_y = -e^x\sin y \quad = -(v_x = e^x\sin y)$$

C.R. is satisfied.

So f is differentiable everywhere.

Polynomials are differentiable

Rational functions are differentible where their denominators are not 0.

C.R. in Polar coordinates:

$$f(r,\theta) = u(r,\theta) + i v(r,\theta)$$

$$r u_r = v_\theta \,, \quad u_\theta = -r v_r$$

$$f'(z_0) = e^{-i\theta_0}(u_r + i v_r)$$ Section 24

Central theme

Def: f is analytic at z_0 if f is differentiable on some ε-neighbhd of z_0.

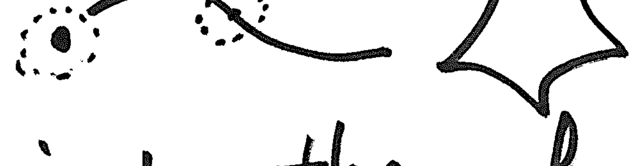
e.g.: $f(z) = |z|^2 = z\bar{z}$ is NOT analytic anywhere.

e^z is analytic everywhere

polynomials are analytic everywhere.

rational functions are analytic where they are defined.

Def: f is analytic on a subset $S \subset \mathbb{C}$ if f is analytic at every point of S.



Remark: If S is open, then f is analytic on S $\Leftrightarrow$ f is differentiable on S

Def: We say f is entire if f is analytic on all of $\mathbb{C}$, i.e., f is differentiable on all of $\mathbb{C}$.

e.g.: polynomials, e^z.

Sums, differences, products, compositions of analytic functions are analytic. the quotient of two analytic functions is analytic where the denominator does not vanish.

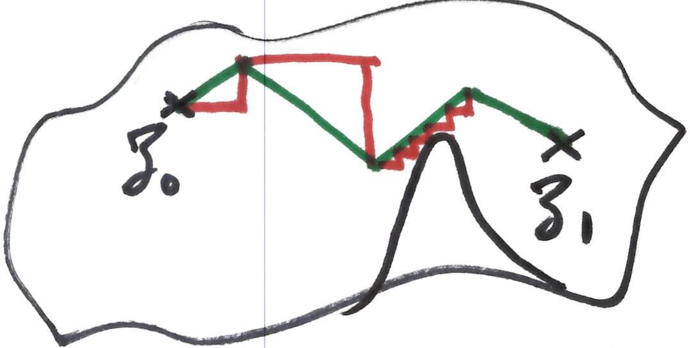
Theorem: If $f'(z) = 0$ on a domain D (open, connected), then f is constant on D.

Proof: $f(z) = u(z) + iv(z)$

$$f'(z) = u_x + i v_x \ \& \ u_x = v_y, u_y = -v_x$$

$$\Rightarrow u_x = 0 = v_x = v_y = u_y$$

$u(x, y)$ $\quad u_x = u_y = 0$ everywhere on D.



u on a horizontal path $u(x, y_2)$

y_2 fixed

derivative $= u_x(x, y_2) = 0$

So u is constant on all horizontal paths.

similarly on a vertical path:

$u(x_2, y)$ $\quad$ derivative $= u_y(x_2, y) = 0$

so u is constant on all vertical paths.

$\Rightarrow$ u is constant.

Similarly v is constant.

So $f = u + iv$ is constant. $\square$

<u>Alternative:</u>

path $\quad z = \alpha(t)$ $\quad z_0 = \alpha(t_0)$

$\quad\quad z_1 = \alpha(t_1)$

$\#$

$u(z) = u(\alpha(t))$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = 0$$

$\Rightarrow$ ~~u is a~~ $u(\alpha(t))$ is a constant function of t $\Rightarrow$ $u(\alpha(t_0)) = u(\alpha(t_1))$

$\Rightarrow$ $u(z_0) = u(z_1)$

<u>Def</u>: We say z_0 is a singular point of f if f is <u>NOT</u> analytic at z_0 but f is analytic at some point of any neighborhood of z_0.

<u>e.g.</u>: - the poles of a rational function are singular points.

- $f(z) = |z|^2 = z\bar{z}$ has NO singular points.

Proposition: If $f(z)$ and $\overline{f(z)}$ are analytic on a domain D, then $f(z)$ is constant.

(equivalently, if f is analytic and non constant on D, then f is NOT analytic on D)

Proof: $f(x,y) = u(x,y) + i v(x,y)$

$$f'(x,y) = u_x + i v_x$$

$$\bar{f} = u - i v \qquad \bar{f}' = u_x - i v_x$$

for f: $u_x = v_y$ & $u_y = -v_x$

for $\bar{f}$: $u_x = -v_y$ & $u_y = -(-v_x)$

$$\Rightarrow \underbrace{v_y}_{= u_x} = -v_y \Rightarrow 0 \ \& \ \underbrace{v_x}_{= u_y} = 0$$

So all partials are 0, similarly to the previous proof, we conclude f = constant.

Prop.: If f is analytic on a domain D and $|f(z)|$ is constant on D, then f is constant on D.

(a priori, the argument of $f(z)$ can change)

Proof: $c = |f(z)|$ $\qquad c^2 = |f(z)|^2$

if $c = 0$, then $f(z) = 0$ is constant.

if $c \neq 0$, then $c^2 = f(z)\overline{f(z)}$

$$\overline{f(z)} = \frac{c^2}{f(z)}$$

$f(z) \neq 0$ because $c \neq 0$

analytic

By the previous proposition, f is constant.

The Laplacian of $u(x,y)$ is

$$\nabla u := u_{xx} + u_{yy} = \frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2}$$

u is harmonic if $\nabla u = 0$

If f is analytic on a domain D, then u & v are harmonic where

$$f = u + iv$$

<u>Later:</u> f is differentiable infinitely many times

So u & v are also differentiable infinitely many times.