Prop.: If f is analytic on a domain D and $|f(z)|$ is constant on D, then f is constant on D.

(a priori, the argument of $f(z)$ can change)

Proof: $c = |f(z)|$ $\quad c^2 = |f(z)|^2$

if $c = 0$, then $f(z) = 0$ is constant.

if $c \neq 0$, then $c^2 = f(z)\overline{f(z)}$

$$\overline{f(z)} = \frac{c^2}{f(z)}$$

$f(z) \neq 0$ because $c \neq 0$

analytic

By the previous proposition, f is constant.

$u(x,y)$ $\quad u$ twice differentiable with continuous partials

The Laplacian of u is

$$\nabla u := u_{xx} + u_{yy} = \frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2}$$

u is harmonic if $\nabla u = 0$

If f is analytic on a domain D, then u & v are harmonic where

$$f = u + iv$$

<u>Later:</u> f is differentiable infinitely many times on D

So u & v are also differentiable <u>infinitely</u> many times.

$$u_x = v_y \quad \& \quad u_y = -v_x \quad (\text{C.R.})$$

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}$$

continuity of partials.

$$\Rightarrow \quad u_{xx} + u_{yy} = 0 = \nabla u$$

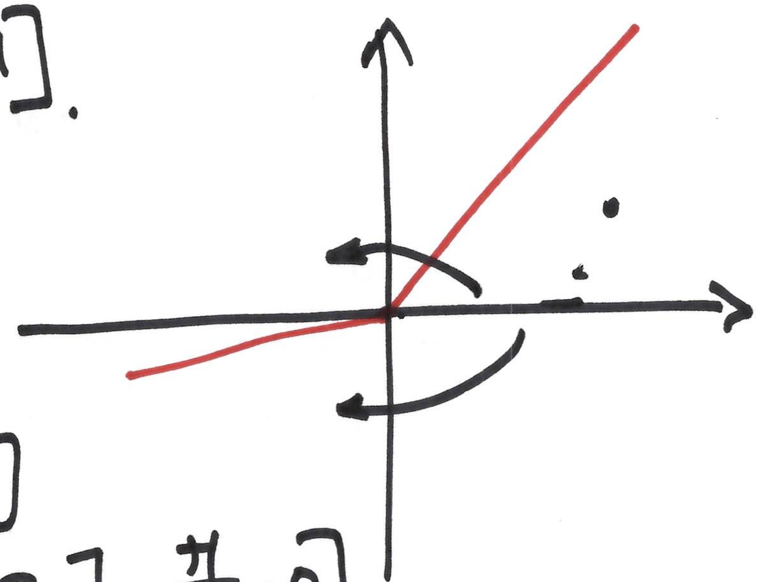
Similarly $\nabla v = 0$ $\square$

<u>Theorem:</u> If u is differentiable twice & its partial derivatives are continuous, then $u_{xy} = u_{yx}$. (from Calculus).

Solutions to practice problems:

6) $f(0) = 0 \qquad f(z) = \sqrt{|z|}\, e^{i \operatorname{Arg} z/2}$

a) $\operatorname{Arg} z \in \,]-\pi, \pi]$.



If $\operatorname{Arg} z \in [0, \pi]$,
then $\operatorname{Arg} f(z) \in [0, \frac{\pi}{2}]$

If $\operatorname{Arg} z \in \,]-\pi, 0]$
then $\operatorname{Arg} f(z) \in \,]-\frac{\pi}{2}, 0]$.

If $\operatorname{Arg} z \in \,]-\pi, \pi[$, then
$f(z)$ is continuous.

If we approach a negative real $-r$ from the upper half plane, the limit of $f(z)$ is $\sqrt{r}\, e^{i\pi/2} = i\sqrt{r}$.

If we approach $-r$ from the lower half plane, the limit of $f(z)$ is
$$\sqrt{r}\, e^{-i\pi/2} = -i\sqrt{r}.$$

So $\lim\limits_{z \to -r} f(z)$ does <u>not</u> exist.

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \sqrt{|z|}\, e^{i\,\mathrm{Arg}\,z/2} \ ?$$

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \left|\sqrt{|z|}\, e^{i\,\mathrm{Arg}\,z/2}\right|$$

$$= \lim_{z\to 0} \sqrt{|z|} = 0$$

$$\Rightarrow \lim_{z\to 0} f(z) = 0$$

<u>Conclusion</u>: f is continuous everywhere except at negative real numbers.

$\mathrm{Arg}\,z$, e^z, $\sqrt{\ }$, $|\ |$ are all continuous when $\mathrm{Arg}\,z \in \left]-\pi, \pi\right[$.

So f is continuous when $\mathrm{Arg}\,z \in \left]-\pi, \pi\right[$.

b) $f(z) = \sqrt{|z|}\, e^{i\,\mathrm{Arg}\,z/2}$.

$\mathrm{Arg}\,z/2$ takes all values in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$|f(z)|$ takes all nonnegative values.

The image of f is the right half plane minus the negative imaginary numbers, meaning $-ri$ $\quad r>0$

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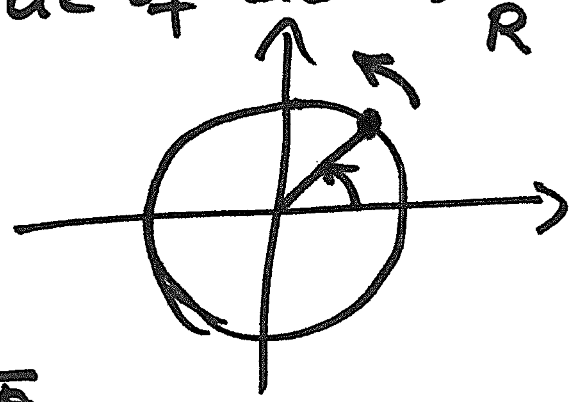
8) $f(z) = \dfrac{1}{\bar{z}} = w.$

(a) $R>0$ $\qquad |z| = R$ circle

$$|z| = R \iff |\bar{z}| = R \iff \left|\frac{1}{\bar{z}}\right| = \frac{1}{R}.$$

$$z = \frac{1}{\bar{w}} = g(w)$$ the inverse of f.

the image is the circle of radius $\frac{1}{R}$ centered at 0.



$z = Re^{i\theta}$ $\quad \theta \nearrow$

$$w = \frac{1}{\bar{z}} = \frac{1}{Re^{-i\theta}}$$

$w = \dfrac{1}{R} e^{i\theta}$ $\quad \theta \to$ w traveling counterclockwise

(b) $\theta_0 \in \mathbb{R}$ $\qquad z = re^{i\theta_0}$ $\quad r \neq 0$

$$w = 1/\bar{z} = \frac{1}{r} e^{i\theta_0}$$

image is the ray with angle θ_0

as $r \nearrow$, $\frac{1}{r} \searrow$

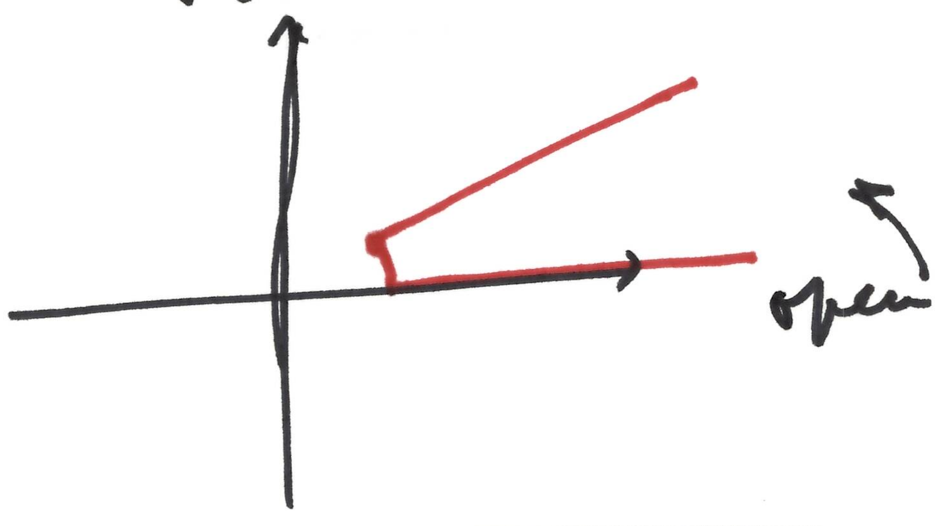
so w travels inward in its ~~pt~~ ray.

(c) $0 < |z| < 2, \quad 0 < \arg z < \pi/6$

from part(a): $|w| = \dfrac{1}{|z|}$ so. $|w| > \frac{1}{2}$

~~$0 < |w| < \frac{1}{2}$~~ from part (b):

$\arg w = \arg z$ so $0 < \arg w < \pi/6$.



(5) $f(z) = z^2 + iz - 3 \qquad |z| \leq 2$

(a) $0 \leq |f(z)| = |z^2 + iz - 3| \leq |z|^2 + |iz| + 3$

$\leq 4 + 2 + 3 = 9$

(b) Can we find z_1 with $|f(z_1)| = 0$

z_2 with $|f(z_2)| = 9$?

$$f(z_1) = 0 = z_1^2 + iz_1 - 3$$

Solve: $z^2 + iz - 3 = 0$

$$z = \frac{-i \pm \sqrt{i^2 - 4(-3)1}}{2} = \frac{-i \pm \sqrt{11}}{2}$$

two possible values for z_1 : are they in the disc $|z| \leq 2$?

$$\left| \frac{-i \pm \sqrt{11}}{2} \right| = \frac{1}{2}\sqrt{1+11} = \frac{1}{2}\sqrt{12}$$

$$= \sqrt{3} \leq 2$$

So both values are in the disc.

$$f(z) = 9e^{i\theta}$$

$$z^2 + iz - 3 = 9e^{i\theta}$$

$$|f(z)| = |z^2 + iz - 3| \leq |z^2| + |iz| + 3$$

$$\leq 9.$$

equality means

$$|z^2 + iz - 3| = |z^2| + |iz| + 3 = 9$$

$|z| = r \qquad r^2 + r + 3 = 9$

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r > 0 \Rightarrow r = 2$

$|z^2 + iz - 3| = 9$

$\cancel{z^2 + iz - 12 = 0}$

$z = 2i \qquad z^2 = -4 \qquad iz = -2$

$z^2 + iz = -6$

$z^2 + iz - 3 = -9$

$0 \leq |a \pm b| \leq |a| + |b| \quad a, b \in \mathbb{C}$

$|a - b| = |a + (-b)|$

$\big||a| - |b|\big| \leq |a \pm b|$

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(4) (b) 5-th roots of -243:

$$-243 = r e^{i\theta}$$

$r = 243$ $\qquad$ $\theta = \pi + 2k\pi.$

$\sqrt[5]{243} = 3$ $\qquad$ $\alpha = \dfrac{\pi}{5} + \dfrac{2k\pi}{5}$

$\cancel{z = 3e^{i\pi/5}}$ $\qquad$ $\cancel{\alpha = 3e}$

$|z| = 3$ $\qquad$ $\arg z = \begin{cases} \dfrac{\pi}{5} \\ \pi/5 + \dfrac{2\pi}{5} \\ \pi/5 + \dfrac{4\pi}{5} \\ \pi/5 + \dfrac{6\pi}{5} \\ \pi/5 + \dfrac{8\pi}{5} \end{cases}$

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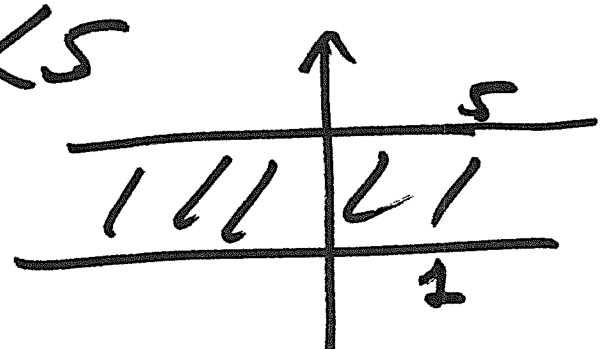
(7) (a) $1 < \operatorname{Im} z < 5$

horizontal strip

not bounded open



(b) $|z + 5i - 1| \le 10$

<u>closed</u> disc centered at $+1 \mp 5i$

with radius 10

bounded

(c) $\mathrm{Re}[(3-2i)z]$ ~~= Re[~~

$z = x+iy \quad (3-2i)z = (3-2i)(x+iy)$

$= 3x+2y+i(-2x+3y)$

$\boxed{3x+2y \leq 7}$

region below the line $3x+2y=7$
closed, not bounded.