<u>Recall:</u> e^z defined on $\mathbb{C}$
differentiable everywhere.

$z = x + iy$

$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$e^{z_1+z_2} = e^{z_1} e^{z_2}$

$\frac{de^z}{dz} = e^z$ $\quad$ e^z is entire

$|e^z| = e^x$ $\quad$ $\arg e^z = y + 2k\pi$

Define a complex logarithm?
$\log z$ should be the inverse function of e^z.

$e^0 = 1 = e^{2\pi i} = e^{4\pi i} = \cdots\cdots$

$w = \log z$ $\quad$ $e^w = z$.

$w = \operatorname{Re} w + i \operatorname{Im} w$

$e^w = e^{\operatorname{Re} w + i \operatorname{Im} w} = e^{\operatorname{Re} w} e^{i \operatorname{Im} w} = z$

$e^{\operatorname{Re} w} = |z|$, $\quad$ $\operatorname{Im} w + 2k\pi = \arg z$

$\operatorname{Re} w = \ln|z| \quad \operatorname{Im} w \text{ "="} \arg z$

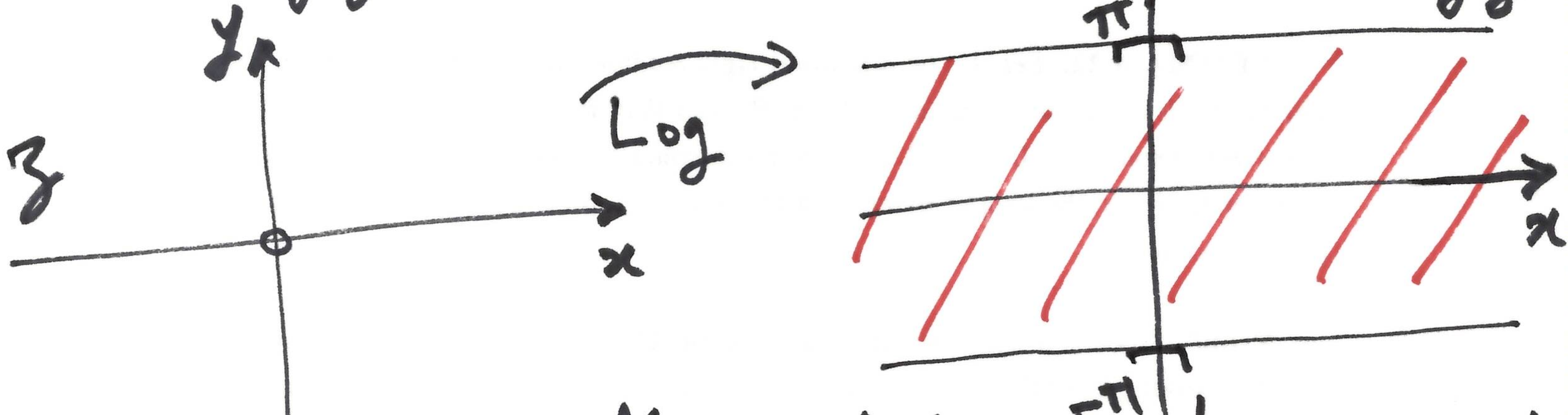
$w = \ln|z| + i \text{ "}\arg z\text{"}$

$\boxed{\text{"}\log z\text{"} = \ln|z| + i \arg z}$ a multi-valued function

for $z \neq 0$

$\boxed{\operatorname{Log} z := \ln|z| + i \operatorname{Arg} z}$

$\operatorname{Arg} z \in \,]-\pi, \pi]$



can choose different intervals for argument

e.g. $\arg z \in [0, 2\pi[$

<u>Differentiability?</u> $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$

$= u + iv$

$z_0 \in \mathbb{C} \setminus \{0\}$.

$\cancel{u = \ln\sqrt{x^2+y^2}}$ $\quad z = re^{i\theta}$ $\quad u = \ln r$, $v = \theta$

$u = \ln r$ | differentiable as a function of r & θ everywhere $(\theta \neq \pi)$ & $r \neq 0$

$v = \theta$ 〃 everywhere

$u_r = \frac{1}{r}$ continuous when $r \neq 0$

$u_\theta = 0$ 〃

$v_r = 0$ 〃

$v_\theta = 1$ 〃

<u>Cauchy-Riemann:</u> $r u_r = v_\theta$?

$u_\theta = -r v_r$?

$r u_r = r \frac{1}{r} = 1 = v_\theta$ ✓

$u_\theta = 0 = -r v_r$ ✓

So $\operatorname{Log} z$ <u>is</u> <u>differentiable</u> with derivative

$$\frac{d}{dz} \operatorname{Log} z = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta} \frac{1}{r} = \frac{1}{r e^{i\theta}} = \frac{1}{z}$$

<u>Rem:</u> $w = \operatorname{Log} z$ $\quad e^w = z$

$e^{\operatorname{Log} z} = z$ take derivatives and use chain rule:

$$e^{\operatorname{Log} z} \cdot \frac{d}{dz} \operatorname{Log} z = 1$$

$$z \quad \frac{d}{dz}\mathrm{Log}\, z = 1$$

$$\frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z}.$$

$$e^{\mathrm{Log}\, z} = z$$

$$\mathrm{Log}(e^z) = \ln|e^z| + i\,\mathrm{Arg}(e^z)$$

$\cancel{z = x+iy}$ e.g. $\boxed{z = -i\pi}$

$$e^z = e^{-i\pi} = -1 \qquad |e^z| = 1$$

$$\mathrm{Arg}\, e^z = \pi$$

$$\mathrm{Log}\, e^z = \ln 1 + i\pi = \underline{i\pi}$$

$$\log z = \ln|z| + i \arg z$$

$$e^{\log z} = e^{\ln|z| + i\arg z} \overset{*}{=}$$

$$= e^{\ln|z|}\cdot e^{i\arg z} = |z|e^{i\arg z}$$

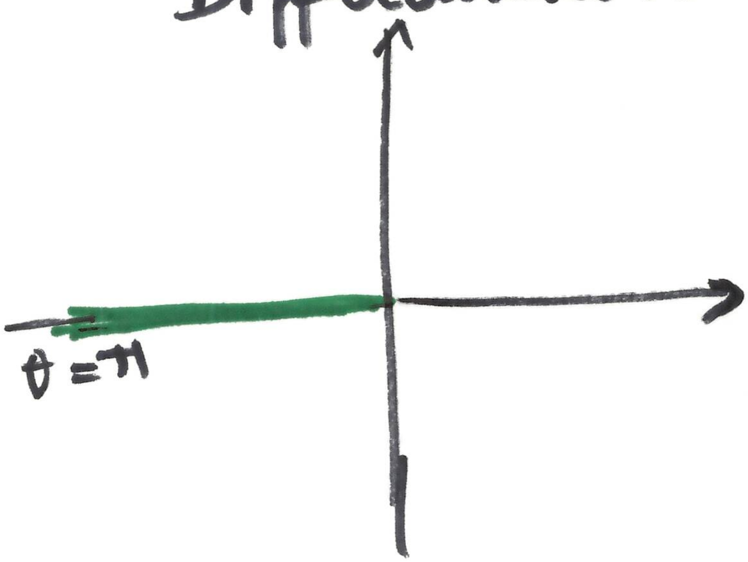
$$= z$$

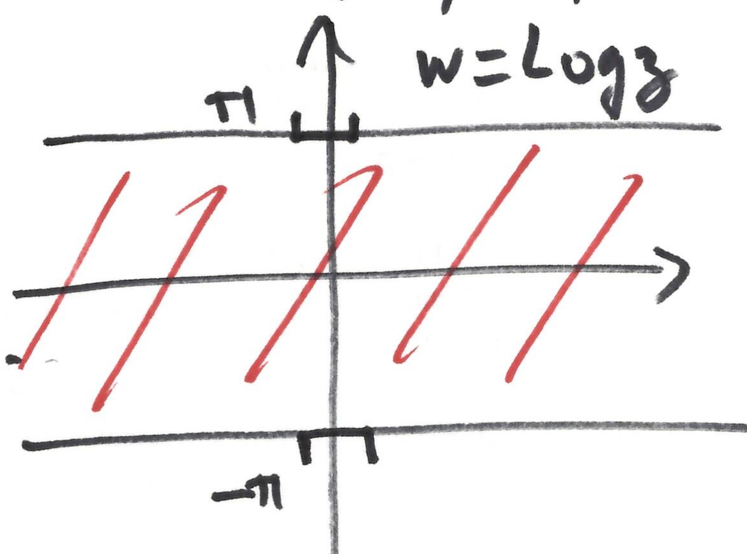
$$z = x+iy, \quad e^z = e^x e^{iy}$$

$$\log e^{z} = \log(e^{x}\cdot e^{iy}) =$$
$$= \ln|e^{x}e^{iy}| + i\arg(e^{x}e^{iy})$$
$$= \ln e^{x} + i\arg(e^{iy})$$
$$= x + i(y + 2k\pi)$$
$$= z + (i\,2k\pi)$$

$$\log z = \ln|z| + i\arg z$$
$$\operatorname{Log} z = \ln \underbrace{|z|}_{r} + i\,\underbrace{\operatorname{Arg} z}_{\theta}$$

Differentiable where $r \neq 0,\ \theta \neq \pi$





remove ray $\theta = \pi$:
branch cut.

open strip.

Log is ~~df~~ Analytic on the domain

$\mathbb{C} \setminus \{\operatorname{Re} z \leq 0, \operatorname{Im} z = 0\}$.

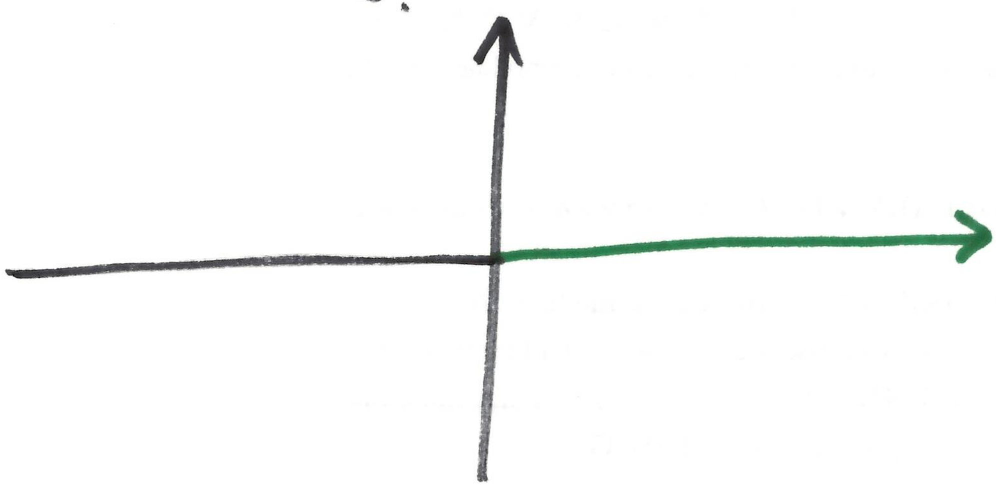
---

e.g. choose $\theta \in]0, 2\pi[$

remove $\operatorname{Re} z \geq 0$

we have what we call a different branch of log ~~ana~~ analytic on

$\mathbb{C} \setminus \{\operatorname{Re} z \geq 0, \operatorname{Im} z = 0\}$



---

e.g. : $]1, 1+2\pi[$ remove ray at $\theta = 1$

z-plane.

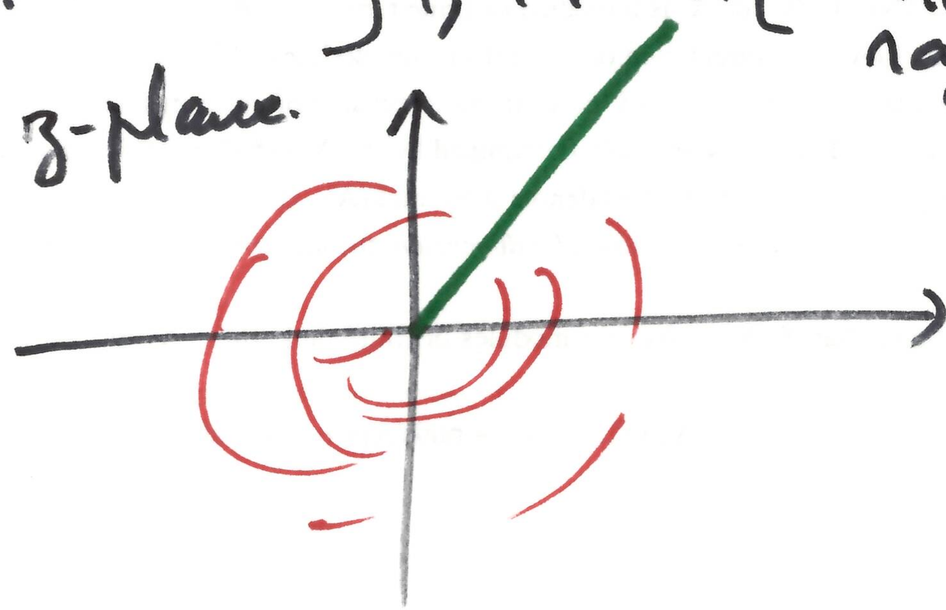
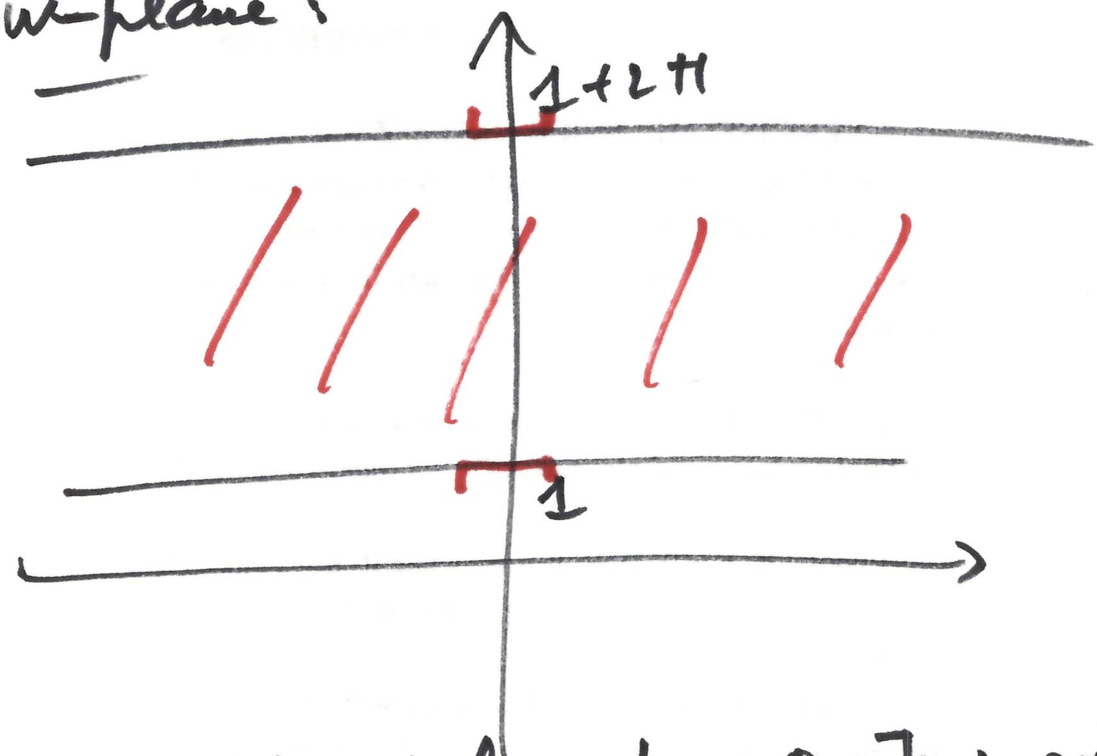


image: w-plane:



Function is well-defined on $\theta \in]1, 1+2\pi]$ but not analytic when $\theta = 1+2\pi$.

Each time we choose one of these intervals, we have a **branch** of the log, the [ray / curve] where the branch is discontinuous is the **branch cut**.

0 belongs to all the branch cuts.
it is called a **branch point**.

$$\ln(x_1 x_2) = \ln x_1 + \ln x_2.$$

$$\mathrm{Log}(z_1 z_2) \stackrel{?}{=} \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$$

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

$$\operatorname{Log}(z_1 z_2) = \ln|z_1 z_2| + i \operatorname{Arg}(z_1 z_2)$$
$$= \ln|z_1| + \ln|z_2| + i \operatorname{Arg}(z_1 z_2)$$

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

when $\operatorname{Arg} z_1, \operatorname{Arg} z_2 \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$

not true e.g. if $\operatorname{Arg} z_1 = \frac{3\pi}{4}$, $\operatorname{Arg} z_2 = \frac{\pi}{2}$

But: $\boxed{\log(z_1 z_2) = \log z_1 + \log z_2}$ as sets

$$\log(z_1 z_2) = \ln|z_1 z_2| + i \arg(z_1 z_2)$$

S_1, S_2 sets of complex numbers:

$$S_1 + S_2 := \{ z_1 + z_2 \mid z_1 \in S_1, z_2 \in S_2 \}.$$

example: $z = i \qquad z^2 = -1$

$$\log i = \ln|i| + i \arg i = i\left(\frac{\pi}{2} + 2k\pi\right)$$

$$\log(i^2) = \log(-1) = \ln|-1| + i \arg(-1)$$
$$= i(\pi + 2k\pi)$$

$$\log i + \log i = i\frac{\pi}{2} + i2k\pi + i\frac{\pi}{2} + i2m\pi$$

$$= i\pi + i(2(k+m)\pi)$$
$$= i\pi + i2k\pi$$
$$2\log i = 2i\left(\frac{\pi}{2} + 2k\pi\right) = i(\pi + 4k\pi)$$
So $\log(i^2) = \log i + \log i$
$$\neq 2\log i$$

<u>More generally</u>: $\log(z_1 z_2) = \log z_1 + \log z_2$

$$\log(z_1 z_2) = \ln|z_1 z_2| + i\arg(z_1 z_2)$$
$$= \ln|z_1| + \ln|z_2| + i\arg(z_1 z_2)$$
$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$$
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$
$$\arg(z_1 z_2) = \theta_1 + \theta_2 + 2k\pi$$
$$\arg z_1 = \theta_1 + 2k\pi$$
$$\arg z_2 = \theta_2 + 2m\pi.$$
$$\log(z_1 z_2) = \ln|z_1| + \ln|z_2| + i(\theta_1 + \theta_2 + 2k\pi)$$
$$\log z_1 + \log z_2 = \ln|z_1| + i(\theta_1 + 2k\pi)$$
$$+ \ln|z_2| + i(\theta_2 + 2m\pi)$$

$$= \ln|z_1| + \ln|z_2| + i(\theta_1 + \theta_2 + 2(k+m)\pi)$$

$$= \log(z_1 z_2)$$

We can use log to define z^c for any $c \in \mathbb{C}$.

$$z^c := e^{c \log z}$$

multi-valued function

$$z^c = e^{c \log z} = \exp(c(\ln|z| + i \arg z))$$

Digression: $\log z^2 \neq 2 \log z$

but $e^{\log z^2} = z^2 = e^{2 \log z}$
