$= \ln|z_1| + \ln|z_2| + i(\theta_1 + \theta_2 + 2(k+m)\pi)$

$= \log(z_1 z_2)$

We can use log to define z^c for any $c \in \mathbb{C}$.

$$z^c := e^{c \log z}$$

<u>multi-valued function</u>

$$z^c = e^{c \log z} = \exp(c(\ln|z| + i \arg z))$$

<u>Digression:</u> $\log z^2 \neq 2 \log z$

but $e^{\log z^2} = z^2 = e^{2 \log z}$ single valued

$\log z = \ln|z| + i \arg z.$

$z^c = \exp(c \ln|z| + i c \arg z)$

$= e^{c \ln|z|} \, e^{i c \arg z}$

$\log z^2 \cancel{\neq} = \log z + \log z$ as sets.

$$e^{\log z^2} = e^{\log z + \log z} = e^{\log z} \cdot e^{\log z}$$

$= e^{\ln|z|} e^{i\arg z} e^{\ln|z|} e^{i\arg z}$

$\cancel{= e^{2\ln|z|} e^{2i\arg z}}.$

$\cancel{= e^{2(\ln|z| + i\arg z)}}.$

$= e^{2\ln|z|} e^{i(\arg z + \arg z)}$

$e^{2\log z} = e^{2\ln|z| + 2i\arg z}$

$= e^{2\ln|z|} e^{2i\arg z}$

$\arg z = \theta + 2k\pi \qquad \arg z + \arg z = \theta + \theta + 2k\pi + 2m\pi.$

$\arg z + \arg z = 2\theta + 2k\pi.$

$2\arg z = 2\theta + 4k\pi.$

$e^{i(\arg z + \arg z)} = e^{i(2\theta + 2k\pi)} = e^{2i\theta}$

$e^{i(2\arg z)} = e^{i(2\theta + 4k\pi)} = e^{2i\theta}$

$\Rightarrow$ z^2 is single valued.

Back to $z^c = \exp(c\log z)$

$= \exp(c\ln|z| + ic\arg z)$

c real $\quad z^c = |z|^c e^{ic\arg z}$

e.g.: $\arg z = \{\theta, \theta+2\pi, \theta+4\pi, \dots, \theta-2\pi, \dots\}$

$\arg z + \arg z = \{2\theta, \theta+\theta+2\pi, \theta+\theta-2\pi, \dots\}$

$2\arg z = \{2\theta, 2\theta+4\pi, 2\theta-4\pi, \dots\}$

e.g. $S = \{0, 1\}$

$S+S = \{0, 1, 2\}$

$2S = \{0, 2\}$ $\quad S+S \neq 2S.$

$$z^c = e^{c\ln|z|}\, e^{ic\arg z}$$

$$z^{c_1+c_2} \overset{?}{=} z^{c_1} z^{c_2}$$ sometimes, not always.

$$z^{c_1-c_2} \overset{?}{=} \frac{z^{c_1}}{z^{c_2}}$$ "

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e.g. $$i^i = e^{i\ln|i| + i\cdot i\cdot \arg i}$$

$$= e^{-\arg i} = e^{-\frac{\pi}{2}+2k\pi}$$

$$= \underbrace{e^{-\pi/2}}_{\text{principal value}} \cdot e^{2k\pi} \quad k\in\mathbb{Z}$$

<u>recall:</u> $e^z = e^{x+iy} = \boxed{e^x(\cos y + i\sin y)}$

Fix a branch of z^c: (fix a branch of arg)

$$z^c = e^{c\ln|z| + ic\,\underline{\arg z}}$$

$\arg z \in \left]\alpha_0, \alpha_0 + 2\pi\right]$. continuous.

differentiable where $\arg z \in \left]\alpha_0, \alpha_0+2\pi\right[$.

($\ln|z|$ continuous.

partial derivatives with respect to x & y exist when $|z| \neq 0$.

~~as~~. C.R. $\Rightarrow z^c$ differentiable.

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c\log z} = e^{c\log z} \cdot \frac{d}{dz}(c\log z)$$

$$= z^c c \frac{1}{z} \cancel{= c z^{c-1}}$$

$$= z^c c z^{-1} = c\, e^{c\log z} e^{-\log z}$$

$$= c\, e^{(c-1)\log z} = c z^{c-1}$$

$[c\log z - \log z = (c-1)\log z]$

because we have chosen one branch, meaning one value for $\log z$.

$$\frac{d}{dz}(c^z) = \frac{d}{dz} e^{z\log c} = e^{z\log c} \frac{d}{dz}(z \log c)$$

$$= e^{z\log c} \log c = \log c \cdot c^z$$

$\log e = \ln|e| + i \arg e = 1 + \underbrace{i \arg e}$.

Choose $\log e = 1$

Is. $\log z^2$ differentiable?

Yes, if we choose a branch, in particular cut out a ray from $\mathbb{C}$. ($z^2 \notin$ ray)

$e^{i\theta} = \cos\theta + i\sin\theta$.

$e^{-i\theta} = \cos\theta - i\sin\theta$.

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Define $\boxed{\cos z = \dfrac{e^{iz} + e^{-iz}}{2}, \ \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}}$

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$ $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$\boxed{\cosh z = \dfrac{e^{z} + e^{-z}}{2} \quad \sinh z = \dfrac{e^{z} - e^{-z}}{2}}$

All entire functions.

$$\sin^2 z + \cos^2 z = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2$$

$$= \frac{e^{2iz}-2+e^{-2iz}}{-4} + \frac{e^{2iz}+2+e^{-2iz}}{4}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$|\sin z| = \left|\frac{e^{iz}-e^{-iz}}{2i}\right| = \frac{|e^{iz}-e^{-iz}|}{2}$$

$$z = -2i \qquad = \frac{|e^{2}-e^{-2}|}{2} = \frac{1}{2}\left|e^2 - \frac{1}{e^2}\right|$$

$z = -Ri \quad R \gg 0 \quad |\sin z|$ gets very large

$$\boxed{\cosh(iz) = \cos z \qquad \sinh(iz) = i\sin z}$$

$$\cos(iz) = \cosh z \qquad \sin(iz) = -\frac{\sinh(z)}{i}$$

$$\boxed{\cos(iz) = \cosh z \qquad \sin(iz) = i\sinh(z)}$$

$$\cosh^2 z - \sinh^2 z = \cos^2(iz) - \left(\frac{1}{i}\sin(iz)\right)^2 = 1$$

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)$$

$$= \frac{1}{2i}\left(i e^{iz} - (-i) e^{iz}\right)$$

$$= \frac{1}{2}\left(e^{iz}+e^{-iz}\right) = \cos z.$$

$$\frac{d}{dz}\left(\sinh(z)\right) = \frac{d}{dz}\left(-i\sin(iz)\right)$$

$$= (-i)\, i \cos(iz) = \cosh z.$$

<u>Zeros of $\sin x$</u> : $x = k\pi$

<u>Zeros of $\sin z$?</u> $\sin z = 0.$

$$\frac{e^{iz}-e^{-iz}}{2} = 0 \qquad \left(e^{iz} = e^{-iz}\right)e^{iz}$$

$$\Rightarrow e^{2iz} = 1 \Rightarrow \log\left(e^{2iz}\right) = \log 1$$

$$\{2iz + 2\pi i m\} = \{i2k\pi\}.$$

$$\Rightarrow 2iz = 2k\pi i \text{ for some } k\in\mathbb{Z}$$

$$\Rightarrow z \cancel{x} = k\pi \text{ for some } k\in\mathbb{Z}.$$

<u>zeros of sinh:</u> $e^x - e^{-x} = 0 \Rightarrow e^{2x} = 1$

$x \in \mathbb{R}$ $\Rightarrow x = 0$.

$e^z - e^{-z} = 0$ $\quad e^z = e^{-z}$

$e^{2z} = 1 \Rightarrow 2z = 2k\pi i$ for some k

$z = k\pi i$ for some k

<u>zeros of cosh:</u> $e^z + e^{-z} = 0$

$e^z = -e^{-z}$ $\quad e^{2z} = -1$

$e^{i\pi} = \cos \pi + i \sin \pi = -1$

$2z = i\pi + 2k\pi i$ for some k

$z = i\frac{\pi}{2} + k\pi i$ //