zeros of sinh: $e^x - e^{-x} = 0 \Rightarrow e^{2x} = 1$
$x \in \mathbb{R}$ $\Rightarrow x = 0$.

$$e^z - e^{-z} = 0 \quad e^z = e^{-z}$$

$$e^{2z} = 1 \Rightarrow 2z = 2k\pi i \text{ for some } k$$

$$z = k\pi i \text{ for some } k$$

zeros of cosh: $e^z + e^{-z} = 0$

$$e^z = -e^{-z} \quad e^{2z} = -1$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$2z = i\pi + 2k\pi i \text{ for some } k$$

$$z = i\frac{\pi}{2} + k\pi i \quad //$$

---

$\tan z = \dfrac{\sin z}{\cos z}$ analytic where $\cos z \neq 0$

$\tanh z = \dfrac{\sinh z}{\cosh z}$ analytic where $\cosh z \neq 0$

---

Inverse trig. functions:

$w = \sin z$ $\quad z = \arcsin w$.

$= \frac{1}{2i}(\exp(iz) - \exp(-iz))$

$$2iw = e^{iz} - e^{-iz}$$

$$e^{iz} - e^{-iz} - 2iw = 0$$

$$e^{2iz} - 2iw\, e^{iz} - 1 = 0$$

$$(e^{iz})^2 - 2iw\, e^{iz} - 1 = 0$$

quadratic formula: (reduced)

$$e^{iz} = iw + ((iw)^2 + 1)^{1/2}$$ <u>two values.</u>

$$e^{iz} = iw + (1 - w^2)^{1/2}$$ <u>two values</u> for square root.

$$iz = \log\left(iw + (1 - w^2)^{1/2}\right)$$

$$z = -i \log\left(iw + (1 - w^2)^{1/2}\right)$$

multivalued

$$z = -i\left(\ln\left|iw + (1 - w^2)^{1/2}\right| + i\arg\left(iw + (1 - w^2)^{1/2}\right)\right)$$

$$z = \arg\left(iw + (1 - w^2)^{1/2}\right) - i\ln\left|iw + (1 - w^2)^{1/2}\right|$$

$\infty$ly many values

two values

<u>Similarly</u>: $\arctan z = \dfrac{i}{2} \log\left(\dfrac{i+z}{i-z}\right)$

$$\cancel{\mathrm{arcs}}\ \arccos z = -i\log\left(z + (z^2-1)^{1/2}\right)$$

After choosing branches, we can compute derivatives:

$$\frac{d}{dz}(\arcsin z) = \frac{1}{\sqrt{1-z^2}}$$

$$\frac{d}{dz}(\arccos z) = -\frac{1}{\sqrt{1-z^2}}$$

$$\frac{d}{dz}(\arctan z) = \frac{1}{1+z^2} \quad \cdots$$

---

<u>e.g.</u>: $\arccos(2) = -i\log\left(2 + (3)^{1/2}\right)$

$$= -i\left(\ln\left|2 \pm \sqrt{3}\right| + i\arg\left(2 \pm \sqrt{3}\right)\right)$$

$$\arg\left(2+\sqrt{3}\right) - i\ln\left(2+\sqrt{3}\right)$$

$$\arg\left(\cancel{\sqrt{3}-2}\ \ 2-\sqrt{3}\right) - i\ln\left(2-\sqrt{3}\right)$$

$$2k\pi - i\ln\left(2+\sqrt{3}\right)$$

$$2k\pi - i\ln\left(2-\sqrt{3}\right)$$

For a function of a real variable,

$\exists c \in [a,b]$ s.t. $$\frac{f(b)-f(a)}{b-a} = f'(c).$$

NOT true for $f(z)$.

<u>reason</u>: $f(z) = u(z) + i\,v(z)$.

$$f(z(t)) = \underbrace{u(z(t))} + i\,v(z(t)).$$

$\exists\, c_1 \in [a,b]$ s.t. $u'(c_1) = \dfrac{u(b)-u(a)}{b-a}$

$\exists\, c_2 \in [a,b]$ s.t. $v'(c_2) = \dfrac{v(b)-v(a)}{b-a}$

put them together to get

$$f'(c) = \frac{f(b)-f(a)}{b-a} \ ?$$

no because $c_1$ does not have to be equal to $c_2$

So the mean value theorem is NOT true for functions of a complex variable.

## Integrals: ~~over~~ on curves.
(contour integrals).

$$z(t) = (x(t), y(t)) \quad t \in \mathbb{R}$$
$$= x(t) + i y(t). \quad \text{on } t \in [a,b] \subset \mathbb{R}$$

as $t$ varies from $a$ to $b$
$z(t)$ traces a curve in the plane..
integrate $f(z)$ on this curve?
$f(z) = u(z) + i v(z)$.

$$\cancel{\int u(t)} \quad \int u\,dx \quad \int u\,dy \quad \cdots$$

$$\int_C f(z)\,dz\ ? \qquad C = \text{curve.}$$
$$z(t) \qquad dz = z'\,dt$$

$$\int_a^b f(z(t))\, z'(t)\,dt.$$

$$= \int_a^b \big(u(x(t), y(t)) + i v(x(t), y(t))\big)\,(x'(t) + i y'(t))\,dt.$$

$$= \int_a^b (u(t) + i v(t))(x' + i y')\,dt$$

$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (uy' + vx')\,dt.$$

<u>Example:</u> $C$ = curve = circle of radius $R$

$$z(t) = Re^{it} \quad t \in [0, 2\pi].$$

$$f(z) = z^n \qquad n \in \mathbb{Z}.$$

$$\int_0^{2\pi} z^n\,dz = \int_0^{2\pi} R^n e^{int} \cdot Rie^{it}\,dt$$

$$= iR^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt$$

two cases: (1) $n+1=0$ or $n=-1$

then $\displaystyle\int_0^{2\pi} z^n dz = i\cancel{R^{n+1}}\int_0^{2\pi} dt = 2\pi i$

So $\displaystyle\int_0^{2\pi} \frac{dz}{z} = 2\pi i$

(2) $n+1 \neq 0$ or $n \neq -1$

$$\int_0^{2\pi} z^n\,dz = iR^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt$$

$$= iR^{n+1}\left[\frac{1}{i(n+1)}e^{i(n+1)t}\right]_0^{2\pi}$$

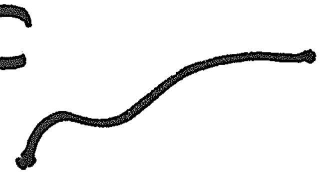
$$= iR^{n+1}\left(\frac{1}{i(n+1)} - \frac{1}{i(n+1)}\right) = 0$$

---

In general, we need $z'(t)$ to exist.
it's OK if $z'$ does not exist at a finite number of points.
What we need is for the curve to be piecewise smooth, meaning a finite union of smooth pieces.
↑ $z'$ exists.

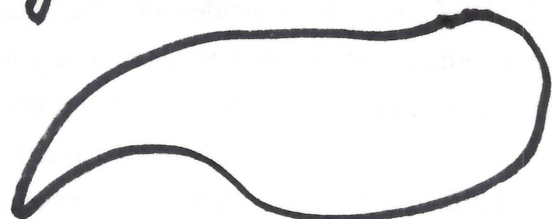
---

We say $C$ is has no self intersection if $z(t_1) \neq z(t_2)$ for all $t_1 \neq t_2$.
i.e., $z : [a, b] \longrightarrow \mathbb{C}$ is 1-to-1.



This is also called a simple arc or a Jordan arc.

If $z(a) = z(b)$ but $z$ is 1-to-1 elsewhere, we say $C$ is a simple closed curve or Jordan curve

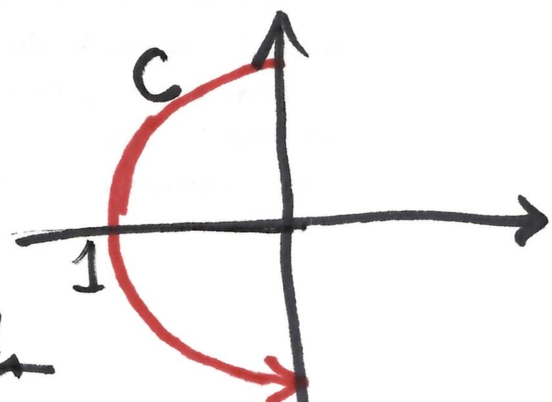


<u>Admit</u>: A simple closed curve $C$ has an interior and an exterior

$$\mathbb{C} \setminus C = \underbrace{\overset{\text{inside}}{\cancel{\text{interior}}}}_{\text{bounded}} \cup \underbrace{\overset{\text{outside}}{\cancel{\text{exterior}}}}_{\text{unbounded}}$$

---

We can integrate continuous functions. At discontinuities, cut the integral into two (or more):



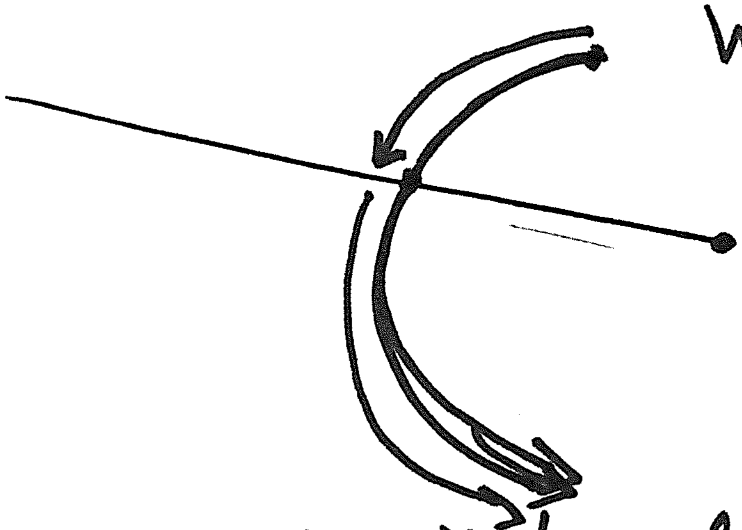

<u>e.g.</u>: $f(z) = z^{1/2}$

one branch
~~principal branch~~

$$z(t) = e^{it} \quad t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$z(t)^{1/2} = e^{it/2} \quad \text{if } t \in \left[\frac{\pi}{2}, \pi\right]$$

We will compute this next time



idea! split integral ~~its~~ into two at the branch cut: compute two integrals and add.

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**Theorem:** $C$ contour: piecewise smooth curve.

$f$ continuous and $M > 0$ real s.t. $|f(z)| \leq M$. on $C$, then

$$\left| \int_C f(z)\, dz \right| \leq ML$$

where $L$ is the length of $C$.

**Proof:** next time.