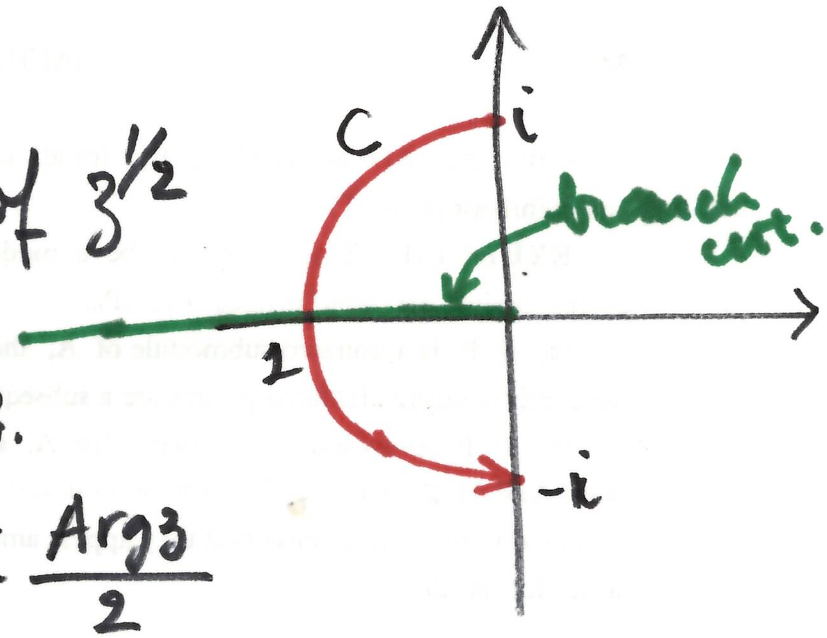
e.g. $f(z) = z^{1/2}$

principal branch of $z^{1/2}$

Arg(z).



$$z = |z| \cdot e^{i \operatorname{Arg} z}.$$

$$z^{1/2} = \sqrt{|z|}\; e^{i \frac{\operatorname{Arg} z}{2}}$$

$$z(\theta) = e^{i\theta} \qquad \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$\operatorname{Arg} z \in \left]-\pi, \pi\right[= (-\pi, \pi)$$

integrate: $\displaystyle\int_{\pi/2}^{3\pi/2} f(z(\theta))\, dz(\theta)$ $\boxed{\begin{aligned} dz &= z' d\theta \\ &= ie^{i\theta}\end{aligned}}$

$$= \int_{\pi/2}^{\pi} \sqrt{|z|}\, i e^{i\theta/2} + \int_{\pi}^{3\pi/2} \sqrt{|z|}\, e$$

e

$$z(\theta) = e^{i\theta} \qquad |z(\theta)| = 1$$

$$z^{1/2}(\theta) = e^{i\theta/2} \quad \text{for } \theta \in \left[\frac{\pi}{2}, \pi\right]$$

$$z^{1/2}(\theta) = e^{i(\theta - 2\pi)/2} \qquad \theta = \left[\pi, \frac{3\pi}{2}\right]$$

$$\int_{\pi/2}^{3\pi/2} z^{1/2} z' \, d\theta = \int_{\pi/2}^{3\pi/2} z^{1/2} i e^{i\theta} \, d\theta$$

$$= i\int_{\pi/2}^{\pi} e^{i\theta/2} e^{i\theta} \, d\theta + i\int_{\pi}^{3\pi/2} e^{i(\theta-2\pi)/2} e^{i\theta} \, d\theta$$

$$= i\int_{\pi/2}^{\pi} e^{\frac{3i}{2}\theta} \, d\theta + i\int_{\pi}^{3\pi/2} e^{\left(\frac{3\theta}{2}-\pi\right)i} \, d\theta$$

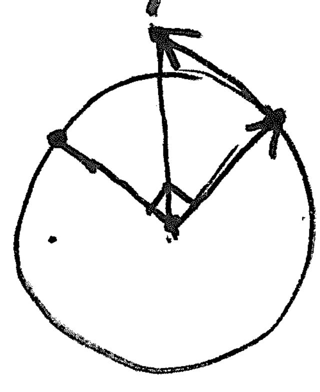
$$= \quad '' \quad + i\int_{\pi}^{3\pi/2} e^{-\pi i} e^{\frac{3\theta}{2}i} \, d\theta.$$

$$= i\left[\frac{1}{3i/2} e^{\frac{3i}{2}\theta}\right]_{\pi/2}^{\pi} - i\left[\frac{1}{3i/2} e^{\frac{3i}{2}\theta}\right]_{\pi}^{3\pi/2}$$

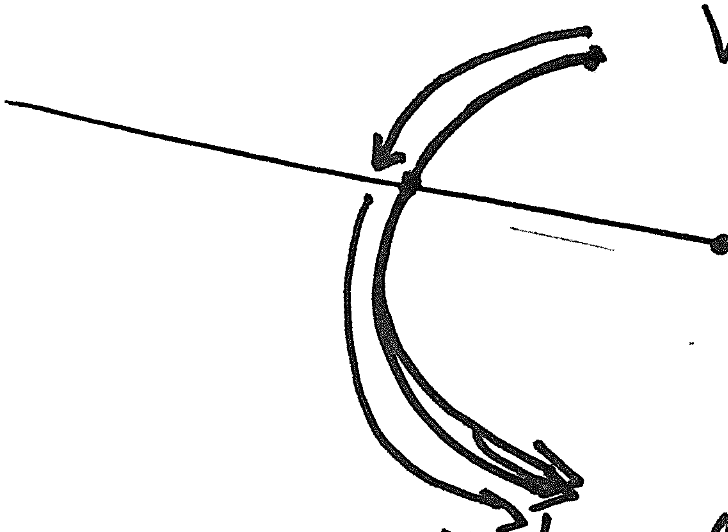
$$= \frac{2}{3}\left(-i - e^{i\frac{3\pi}{4}} - \left(e^{9i\pi/4} - (-i)\right)\right)$$

$$= \frac{2}{3}\left(-2i - e^{3\pi/4\, i} - e^{\pi/4\, i}\right)$$

$$= \frac{2}{3}\left(-2i - \sqrt{2}\, i\right)$$



We will compute this next time



idea! split integral ~~its~~ into two at the branch cut: compute two integrals and add.

Theorem: C contour: piecewise smooth curve.

f continuous. and $M > 0$ real s.t. $|f(z)| \leq M$. on C, then

$$\left| \int_C f(z)\, dz \right| \leq ML$$

where L is the length of C.

Proof: next time.

$z(t) \qquad z : [a,b] \longrightarrow \mathbb{C}$.

z continuous, piecewise smooth.

$z'(t)$ exists except possibly at finitely many points and $z'(t) \neq 0$ except possibly at finitely many points.

$0 \leq |f(z(t))|$ continuous function of a real variable on $[a,b]$.

$\Rightarrow$ $|f(z(t))|$ has a maximum on $[a,b]$.

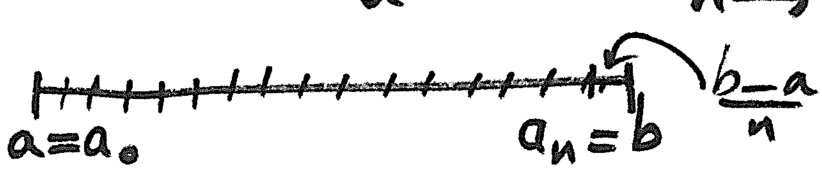
$M \geqslant$ maximum of f on $[a,b]$.

<u>Proof of theorem:</u>

<u>Lemma:</u> For any $w : [a,b] \longrightarrow \mathbb{C}$ continuous.

$$\left| \int_a^b w(t)\,dt \right| \leq \int_a^b |w(t)|\,dt.$$

<u>Proof:</u>
$$\int_a^b w(t)\,dt = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) w(c_i)$$



$c_i \in$ i-th interval.

$$\left|\int_a^b w(t)\,dt\right| = \left|\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) w(c_i)\right|$$

$a_n \to L$. f is continuous.

$$\lim_{n\to} f(a_n) = \cancel{\lim}\, f(L) = f(\lim a_n)$$

$| \ |$ is continuous.

$$= \lim_{n\to+\infty} \left|\sum_{i=1}^{n} \frac{b-a}{n} w(c_i)\right|$$

$$\leq \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{b-a}{n} |w(c_i)|$$

$$= \int_a^b |w(t)|\,dt. \qquad \square$$

Proof of theorem:

$C: \quad z : [a,b] \to \mathbb{C}$

piecewise smooth.

First assume z is smooth

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

apply lemma:

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$$

$$\leq \int_a^b |f(z(t))\,z'(t)|\,dt$$

$$= \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\leq \int_a^b M\,|z'(t)|\,dt$$

$$= M\int_a^b |z'(t)|\,dt$$

$z = (x, y) \qquad z' = (x', y')$

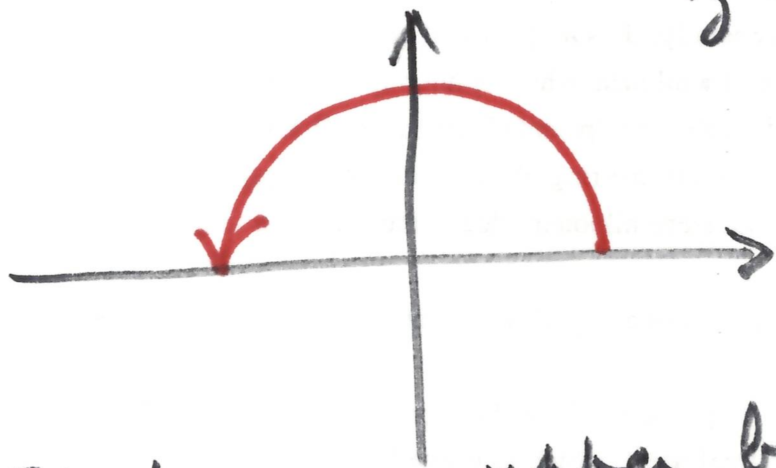
$$|z'| = \sqrt{x'^2 + y'^2}.$$

$$L = (\text{length of } C) = \int_a^b \sqrt{x'^2 + y'^2}\,dt.$$

So

$$\left|\int_C f(z)\,dz\right| \leq ML$$

Example: $f(z) = \dfrac{z+5}{z^3+1}$ C = upper semi-circle of radius R.



Find an upper bound for $|f|$:

$$|z+5| \leq |z| + 5 = R+5$$

$$|z^3+1| \geq \big||z^3| - 1\big| = |R^3 - 1|$$

If $R > 1$: $|f(z)| \leq \dfrac{R+5}{R^3-1}$

$$0 \leq \left| \int_C f(z)\,dz \right| \leq \frac{R+5}{R^3-1} \cdot \pi R \cancel{= \pi R^2+5}$$

$$\lim_{R\to\infty} \pi R \frac{R+5}{R^3-1} = \lim_{R\to\infty} \frac{\frac{\pi}{R} + \frac{5\pi}{R^2}}{1 - \frac{1}{R^3}} = 0$$

$$\Rightarrow \int_C f(z)\,dz \longrightarrow 0 \text{ as } R \to \infty.$$

Anti-derivatives: f continuous function on a domain D, an anti-derivative for f is a function F on D s.t. $F' = f$.

We saw before: anti-derivatives are unique up to addition of a constant.

Theorem: (Fundamental theorem of Calculus)

$f: D \longrightarrow \mathbb{C}$ D domain
continuous.

The following are equivalent:

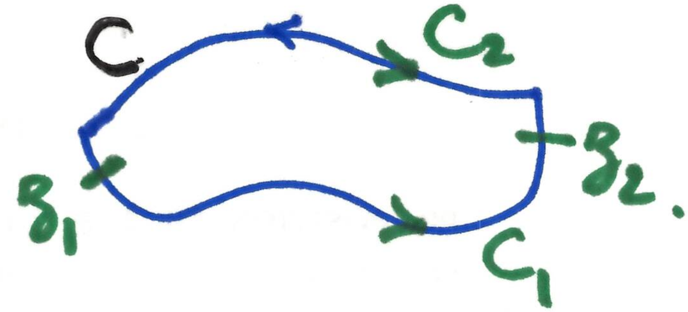
(a) f has an anti-derivative on D.

(b) $\forall\ z_1, z_2 \in D$ and any contour C from z_1 to z_2 in D, $\int_C f\,dz$ is independent of the choice of C.

(c) $\forall$ closed contour $C \subset D$

$$\int_C f\,dz = 0$$

proof: (b) $\Rightarrow$ (c)



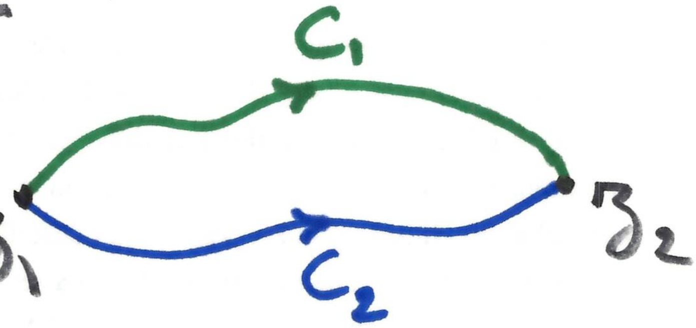
$$C = C_1 - C_2$$

Our $\displaystyle\int_{C_1} f\,dz = \int_{C_2} f\,dz.$

$$\Rightarrow \quad 0 = \int_{C_1} f\,dz - \int_{C_2} f\,dz = \int_{C_1} f\,dz + \int_{-C_2} f\,dz$$

$$= \int_{C} f\,dz$$

(c) $\Rightarrow$ (b)



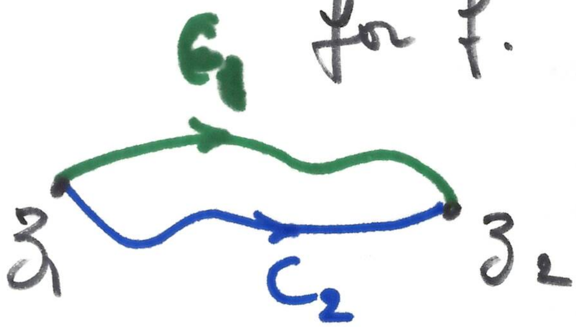
Call: $C = C_2 - C_1$

$$(c) \Rightarrow \int_{C} f\,dz = 0$$

$$\int_{C_2 - C_1} f\,dz = \int_{C_2} f\,dz + \int_{-C_1} f\,dz$$

$$= \int_{C_2} f\,dz - \int_{C_1} f\,dz$$

(a) $\Rightarrow$ (b) Let F be an anti-derivative for f. $F'(z) = f(z)$ on D



$C_1 : z : [a,b] \to D \subset \mathbb{C}$

$C_2 : w : [c,d] \to D \subset \mathbb{C}$

$$\int_{C_1} f\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

$$\left[F(z(t))\right]' = F'(z(t))\, z'(t) = f(z(t))\, z'(t)$$

$$\int_{C_1} f\,dz = \int_a^b \left(F(z(t))\right)' dt$$

$$= F(z(b)) - F(z(a))$$

$$= F(z_2) - F(z_1)$$

Similarly: $\int_{C_2} f\,dz = F(\cancel{z}\, w(d)) - F(w(c))$

$$= F(z_2) - F(z_1)$$

(b) $\Rightarrow$ (a) Fix $z_0 \in D$, for any $z \in D$

$F(z) := \int_C^{\cancel{z}} f(z)\,dz$. C any contour from z_0 to z

$$= \int_{z_0}^{z} f(z)\, dz$$

Prove: $F'(z) = f(z)$ on D