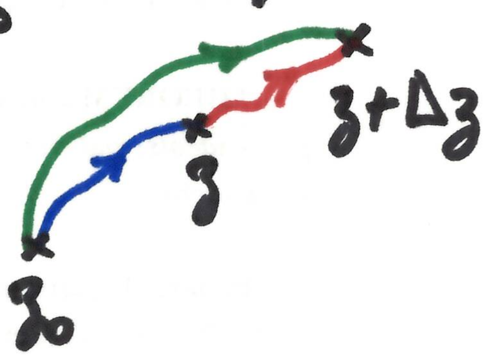
$$= \int_{z_0}^{z} f(z)\,dz$$

Prove: $F'(z) = f(z)$ on D

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} \quad ?$$

$$\frac{1}{\Delta z}\left(\int_{z_0}^{z+\Delta z} f(g)\,dg - \int_{z_0}^{z} f(g)\,dg \right)$$

$$= \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(g)\,dg.$$



Want to show $\lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(g)\,dg = f(z)$

$$\left(\frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(g)\,dg \right) - f(z)$$

$$\int_{z}^{z+\Delta z} 1\,dg = \Delta z$$ because $z' = 1$ (by part(a))

So $f(z) = \frac{1}{\Delta z} \Delta z \, f(z)$

$$f(z) = \frac{f(z)}{\Delta z}\int_z^{z+\Delta z} d\xi = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\, d\xi$$

$$\frac{1}{\Delta z}\int_z^{z+\Delta z} f(\xi)\, d\xi - f(z) =$$

$$\frac{1}{\Delta z}\int_z^{z+\Delta z} \big(f(\xi) - f(z)\big)\, d\xi$$

So

$$\left|\frac{1}{\Delta z}\int_z^{z+\Delta z} \big(f(\xi)-f(z)\big)\, d\xi\right| \leq$$

$$\frac{1}{|\Delta z|}\int_z^{z+\Delta z} |f(\xi)-f(z)|\, d\xi$$

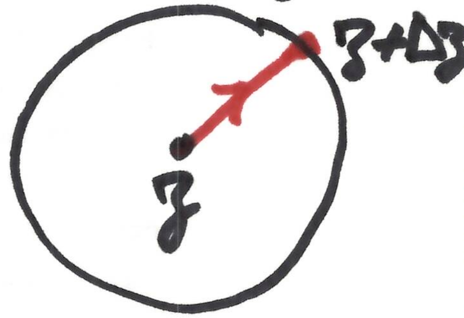
f continuous $\Rightarrow$ $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. ~~for~~ if $|\xi - z| < \delta$, then

$$|f(\xi) - f(z)| < \varepsilon.$$

Choose $|\Delta z|$ very small so that the disc of center z and radius $|\Delta z|$

is contained in D.
Choose the path (contour) from z to $z+\Delta z$ to be the line segment

So $|g-z| < |\Delta z|$



Also Choose $|\Delta z| < \delta$,

then $|g-z| < |\Delta z| < \delta$

$$\Rightarrow \quad |f(g)-f(z)| < \varepsilon.$$

So:

$$\left| \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) \right| =$$

$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(g)-f(z))\, dg \right|$$

$$\leq \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} |f(g)-f(z)|\, dg \right|$$

$$< \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} \varepsilon \, dg \right| = \frac{\varepsilon}{|\Delta z|} |\Delta z|$$

$$= \varepsilon.$$

So we have shown: $\forall \varepsilon > 0, \ \exists \delta > 0$ s.t. if $|\Delta z| < \delta$, then

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \varepsilon.$$

So $\displaystyle\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$ $\square$

<u>Cauchy-Goursat theorem:</u>

If $f(z)$ is analytic at all the points inside and on a simple closed contour C, then $\displaystyle\int_C f(z)\,dz = 0$

Proof if we assume f' continuous: (due to Cauchy).

$f(z) = u + iv \qquad dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

<u>Green's theorem</u>: If P and Q are functions of x & y with continuous partial derivatives, then

$$\int_C (P\,dx + Q\,dy) = \int_R (Q_x - P_y)\,dx\,dy$$

where R = region bounded by C.

Apply this to $\int_C f(z)\,dz$:

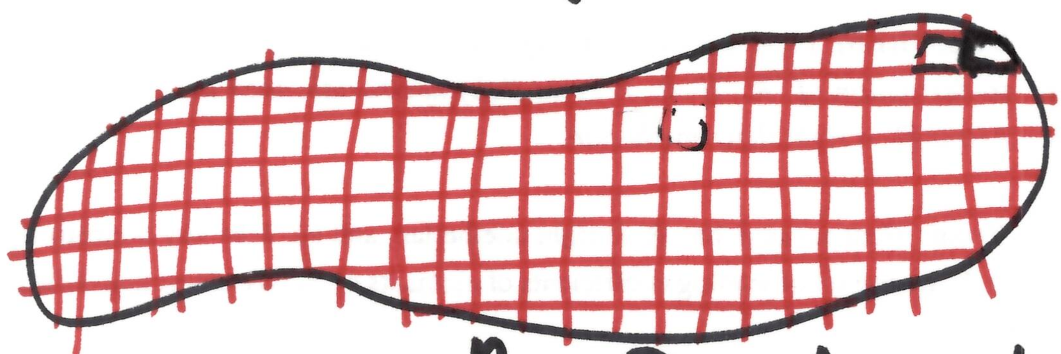
$$\int_C f(z)\,dz = \int_R (-v_x - u_y)\,dx\,dy$$

$$+ i\int_R (u_x - v_y)\,dx\,dy$$

$= 0$ by Cauchy-Riemann

Idea of Goursat's proof for the general case: $\left|\int_C f(z)\,dz\right| \leq \text{Constant } \varepsilon$

$\forall \ \varepsilon > 0$ Constant independent of ε



$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{i\text{th square}} f(z)\,dz$$

$$\left|\int_C f(z)\,dz\right| \leq \sum_{i=1}^{n} \left|\int_{i\text{-th square}} f(z)\,dz\right|$$

$$f(z) = f(z_i) + f'(z_i)(z - z_i) + \delta_i(z)(z - z_i)$$

$z, z_i \in$ i-th square.

$z_i \leftarrow$ fixed.

$$\int_{\text{square}} f(z_i)\,dz = \int_{\text{square}} f(z_i)(z - z_i)\,dz = 0$$

because they have anti-derivatives.

Show $\exists$ good $z_i \in$ i-th square so that $\left| \int_{\text{square}} \delta_i(z)(z - z_i)\, dz \right|$ is small enough.

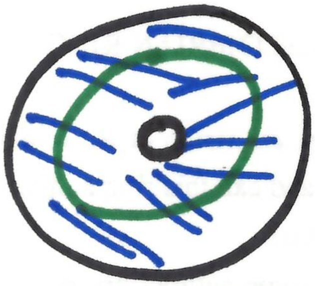
<u>Definition</u>: We say a domain D is simply connected if for every simple closed curve C in D, the interior of C is contained in D.

e.g. - an open disc is simply connected.

- the whole plane is simply connected.
- open rectangles are simply connected.

In other words, every C can be shrunk to a point while staying in D.

"D does not have "holes"."



Not simply connected.

$\mathbb{C} \setminus \{0\}$ not simply connected

$\frac{1}{z}$ $\qquad$ $\mathbb{C} \setminus$ ray simply connected.

So on $\mathbb{C} \setminus \{0\}$ $\frac{1}{z}$ has some nonzero contour integrals.

On $\mathbb{C} \setminus$ ray $\int_C \frac{1}{z}\,dz = 0$ $\quad C \subset \mathbb{C} \setminus$ ray

---

<u>Theorem</u>: If f is analytic on a simply connected domain D, then

$$\int_C f(z)\,dz = 0 \quad \forall \; C \subset D.$$

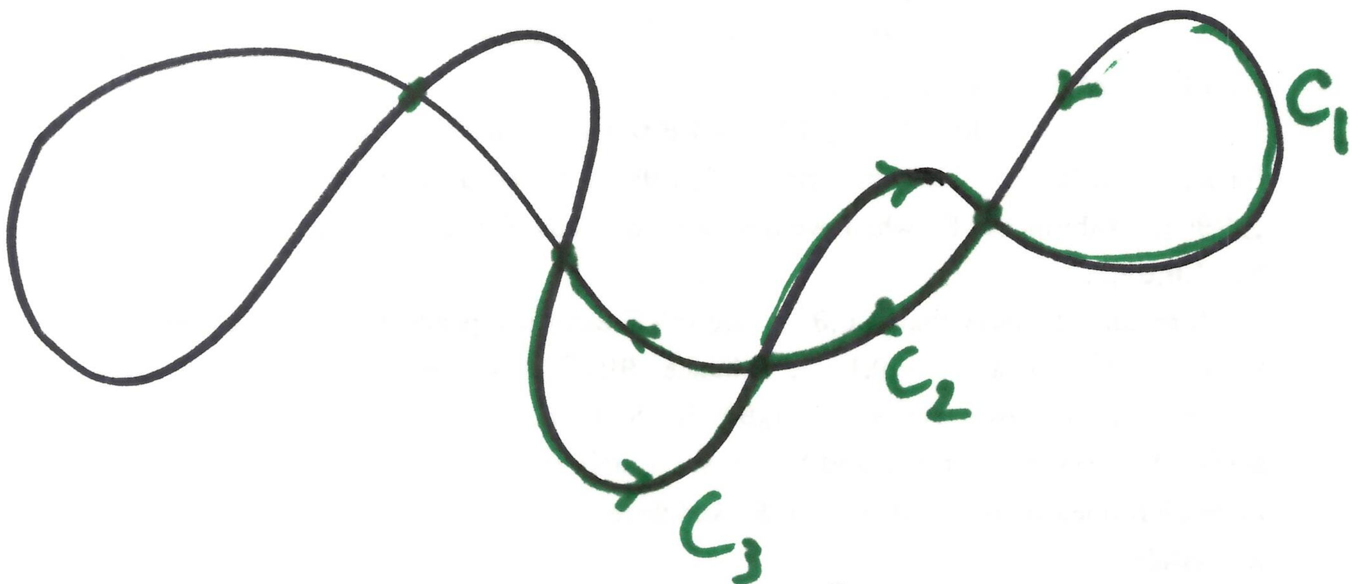
(Fundamental theorem $\Rightarrow$ f has an anti-derivative on D)

<u>Proof</u>: Because D is simply connected $\forall$ simple closed contour $C \subseteq D$ interior of C

is contained in D. So f is analytic on C and inside C

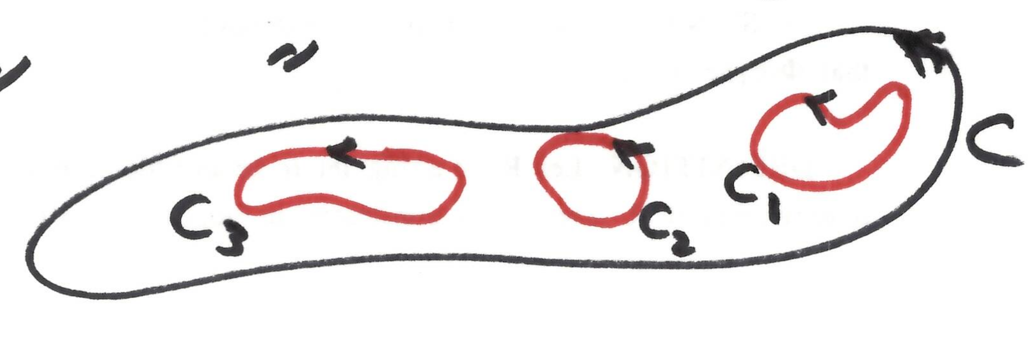
Cauchy-Goursat $\Rightarrow \int_C f(z)\,dz = 0.$ $\square$

<u>Remark:</u> Theorem work for any closed contour:



$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 0$$

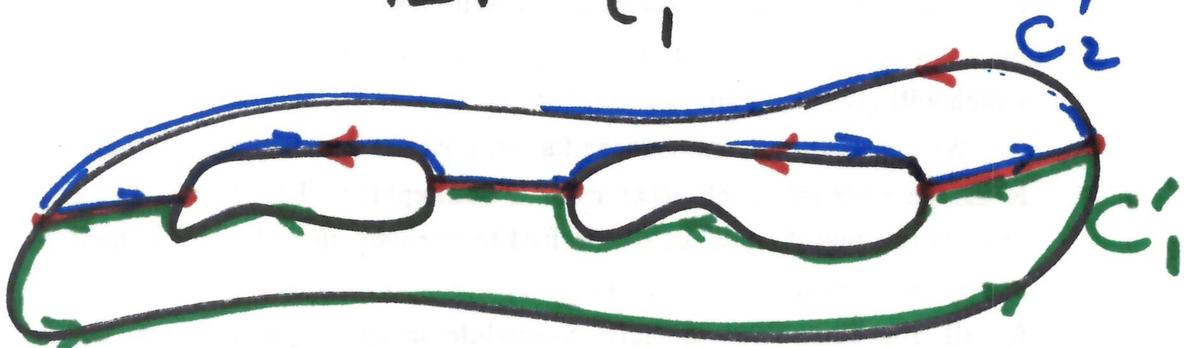
<u>Practical use:</u> Suppose C is a simple closed contour containing $C_1, \dots, C_n$



<u>Corollary</u>: If f is analytic on C, all C_i and on the whole region between them, then

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz.$$

<u>Proof</u>:



$$\int_C f(z)\,dz - \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = \int_{C'_1} f(z)\,dz + \int_{C'_2} f(z)\,dz$$

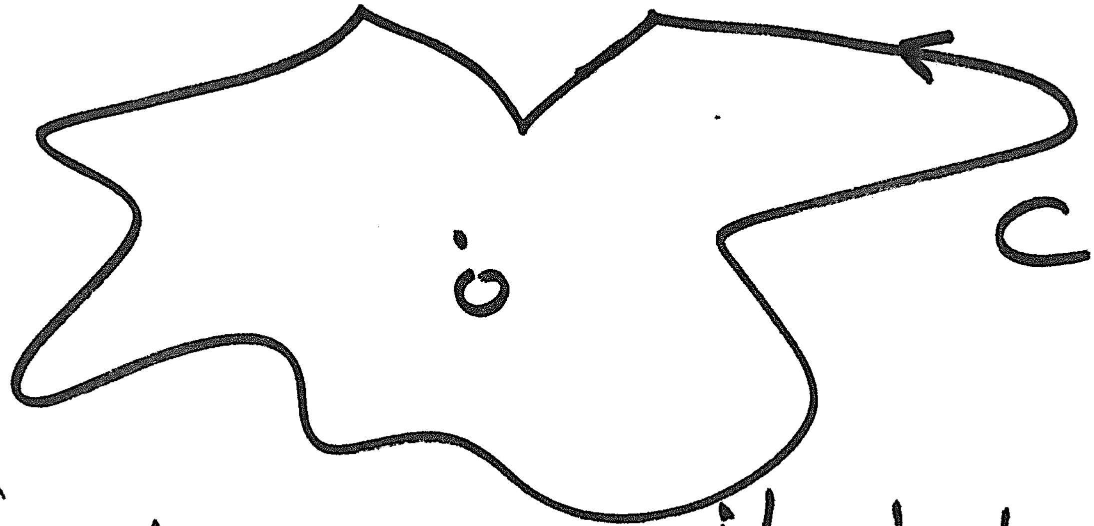
$$= 0$$

<u>e.g.</u> $\int_C z^n\,dz$ $\quad C =$ circle of radius R

$= 0$ if $n \neq -1$

$= 2\pi i$ if $n = -1$

$\Rightarrow$



$$\int_C z^n dz = 0 \quad \text{if } n \neq -1$$
$$= 2\pi i \quad \text{if } n = -1$$

Cauchy integral formula:

C simple closed contour.
f analytic on C and inside C.
z_0 point inside C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

More generally: f is in fact differentiable to all orders and $\forall\, n \geq 0$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Example: $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$ planetary motion.

$0 < a < 1$

$z = e^{i\theta}$

$dz = i e^{i\theta}\, d\theta$

$\quad = iz\, d\theta$

$\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$\quad = \dfrac{z+z^{-1}}{2}$

$$\int_{\text{circle}} \frac{dz}{iz\left(a\frac{z+z^{-1}}{2}z+1\right)} = \cancel{\int_{\text{circle}} \frac{2i\,dz}{iz^2+1}}$$

$$= \int_{\text{circle}} \frac{-2i\,dz}{z\left(az+az^{-1}+2\right)}$$