Example: $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$ planetary motion.

$0<a<1$

$z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$= iz\,d\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$= \frac{z+z^{-1}}{2}$

$$\int_{\text{circle}} \frac{dz}{iz\left(a\frac{z+z^{-1}}{2}+1\right)} = \cancel{\int_{\text{circle}} \frac{2i\,dz}{z^2+1}}$$

$$= \int_{\text{circle}} \frac{-2i\,dz}{z\left(az+az^{-1}+2\right)}$$

$0<a<1$

$az+az^{-1}+2 = 1+a\cos\theta$ never 0.

$$= \int_{\text{circle}} \frac{-2i\,dz}{az^2+2z+a}$$

$0<a<1$

$az^2+2z+a=0$ when $z = \frac{-1\pm\sqrt{1-a^2}}{a}$

$z = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2}-1}$

$\sqrt{\frac{1}{a^2}-1} < \frac{1}{a}$

$$z_1 = -\frac{1}{a} - \sqrt{\frac{1}{a^2}-1} < -1 \quad \text{\underline{not} in the unit disc.}$$

$$z_2 = -\frac{1}{a} + \sqrt{\frac{1}{a^2}-1}$$

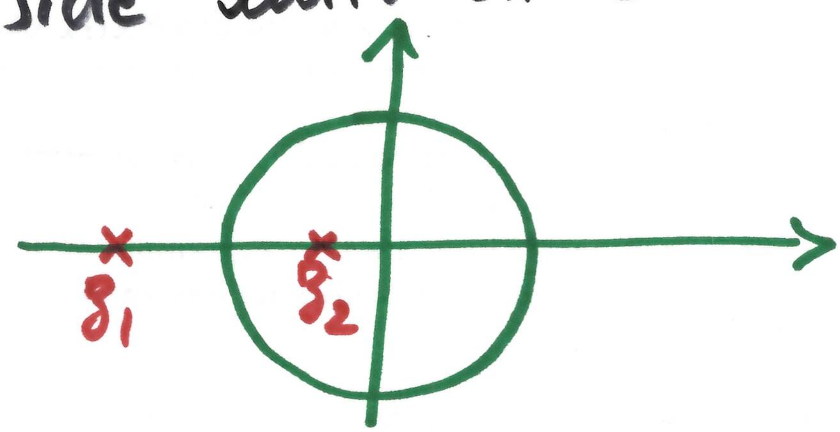
$$\sqrt{1+x} \underset{x \text{ small}}{\cong} 1 + \frac{x}{2}.$$

$$\frac{\sqrt{1-a^2}}{a} \approx \frac{1-\frac{a^2}{2}}{a} \approx \frac{1}{a} - \frac{a}{2}$$

$$-\frac{1}{a} + \sqrt{\frac{1}{a^2}-1} \approx -\frac{1}{a} + \frac{1}{a} - \frac{a}{2} = -\frac{a}{2}$$

$$-\frac{1}{2} < -\frac{a}{2} < 0$$

inside unit circle



$$az^2 + 2z + a = (z-z_1)(z-z_2)$$

$$\int_{\text{circle}} \frac{-2i\,dz}{(z-z_1)(z-z_2)} \qquad = \frac{2i}{z-z_1} = f(z)$$

f analytic on the circle & inside

so $$\int_{\text{circle}} \frac{-2i\,dz}{(z-z_1)(z-z_2)} = 2\pi i\, f(z_2)$$

$$= 2\pi i \frac{-2i}{z_2 - z_1}$$

So

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{4\pi}{2\sqrt{\frac{1}{a^2}-1}}$$

---

e.g.: ① $\int_C \frac{e^z}{z^2} dz$ $\quad$ C contains 0 inside, $n=1$

$$= 2\pi i \, (e^z)'_{\text{at } z=z_0=0}$$

$$= 2\pi i$$

② $\int_C \frac{\cos z}{z^2} dz$ $\quad$ C with 0 inside

$(\cos z)' = -\sin z$ at 0 $= 0$

$$= 2\pi i \, 0 = 0.$$

③ $\int_C \frac{|z| e^z}{z^2} dz$ $\quad$ 0 inside C

$|z| = \sqrt{z\bar{z}}$ <u>NOT</u> analytic anywhere.

integral depends on C.

e.g. $C =$ circle of radius R centered at 0.

$z = Re^{i\theta}$ $\quad |z| = R$

~~$dz = Rie^{i\theta}d\theta$~~.

$$\int_{\text{circle}} \frac{|z|e^z}{z^2}\,dz = R\int \frac{e^z}{z^2}\,dz = 2\pi i R$$

depends on R.

<u>Proof of the Cauchy integral:</u>

$$f(z_0) = \int_C \frac{f(z)}{z-z_0}\,dz \cdot \frac{1}{2\pi i}$$

C contour, simple closed. $z_0 \in$ inside of C

$$\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0).$$

C_ρ = circle of radius ρ centered at z_0, completely inside C

then $\dfrac{f(z)}{z-z_0}$ analytic on C, C_ρ and between them

so $$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$$

$$\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0)$$

$z = z_0 + e^{i\theta}$

$$\int_{C_\rho} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i e^{i\theta}\,d\theta}{e^{i\theta}} = 2\pi i$$

$$= \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0}$$

$$= \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int_0^{2\pi} i\,\big(f(z)-f(z_0)\big)\,d\theta$$

$z = z_0 + \rho e^{i\theta}$ $\quad dz = \rho i e^{i\theta} d\theta$

So $$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \int_0^{2\pi} |f(z)-f(z_0)|\,d\theta$$

f continuous $\Rightarrow \forall\, \varepsilon > 0,\ \exists\, \delta > 0$

s.t. $|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$

Choose $\rho < \delta$, then $|z-z_0| = \rho < \delta$

So $|f(z)-f(z_0)| < \varepsilon$ on $C_\rho \ni z$

So $$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \int_0^{2\pi} \varepsilon \cdot d\theta = 2\pi\varepsilon$$

The above is true for all $\varepsilon > 0$

So $$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq 2\pi\varepsilon \quad \forall\, \varepsilon > 0$$

$$\Rightarrow \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \quad \square$$

Theoretical consequences of Cauchy's formulas:

Theorem (in fact part of the formulas):

If $f(z) = u(z) + iv(z)$ is analytic at z_0 ($\Rightarrow$ analytic on some ε-neighborhood of z_0)

then f is differentiable infinitely many times at z_0, all the derivatives are also analytic on some neighborhood of z_0. So u & v also have partials to arbitrary order in some neighborhood of z_0.

Morera's theorem: If f is continuous on a domain D and $\int_C f\,dz = 0$

for all closed contours $C \subset D$, then f is analytic on D.

<u>Proof</u>: By the fundamental theorem of Calculus, f has an anti-derivative F on D. $F' = f$ so F is analytic on D. $\Rightarrow F$ has derivatives to arbitrary order, $f = F'$ $f' = F''$, ...
$\Rightarrow f$ ~~also~~ is analytic on D. $\square$

<u>Proof of</u>: $f'(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-z_0)^2}\,dz$

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} =$$

$$= \frac{1}{\Delta z}\cdot\frac{1}{2\pi i}\int_C \frac{f(z+\Delta z)-f(z)}{z-z_0}\,dz$$

Δz small enough

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}.$$

Δz small enough so that $z_0+\Delta z$ is also inside C

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

$$f(z_0+\Delta z)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0-\Delta z}\,dz$$

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=$$

$$\frac{1}{2\pi i\,\Delta z}\int_C f(z)\left(\frac{1}{z-z_0-\Delta z}-\frac{1}{z-z_0}\right)dz$$

$$=\frac{1}{2\pi i\,\Delta z}\int_C f(z)\left(\frac{(z-z_0)-(z-z_0-\Delta z)}{(z-z_0)(z-z_0-\Delta z)}\right)dz$$

$$=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-\Delta z)}\,dz$$

now subtract $\displaystyle\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$

$$\frac{1}{2\pi i}\int_C f(z)\left[\frac{1}{(z-z_0)(z-z_0-\Delta z)}-\frac{1}{(z-z_0)^2}\right]dz$$

$$=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\left(\frac{1}{z-z_0-\Delta z}-\frac{1}{z-z_0}\right)dz$$

$$=\frac{1}{2\pi i}\int_C \frac{f(z)\quad\Delta z}{(z-z_0)^2(z-z_0-\Delta z)}\,dz$$

* Show that this integral goes to 0 when $\Delta z \to 0$.

$d =$ ~~longest~~ ~~di~~ shortest distance between z_0 and points on C

$d > 0$

and $|z-z_0| \geq d$ for z on C

$$|z-z_0-\Delta z| \geq \big||z-z_0|-|\Delta z|\big|$$

$$\geq d-|\Delta z|$$

$\uparrow$ very small

$|f(z)| \leq M$ on C

So the modulus of the integral is at most: (theorem in Section 47)

$$\frac{M\,|\Delta z|}{d^2(d-|\Delta z|)} \cdot \text{length of } C$$

(because $\left|\dfrac{f(z)\,\Delta z}{(z-z_0)^2(z-z_0-\Delta z)}\right| \leq \dfrac{M|\Delta z|}{d^2(d-|\Delta z|)}$)

length of C is fixed and

$$\lim_{\Delta z \to 0} \frac{M\,|\Delta z|}{d^2(d-|\Delta z|)} = 0$$

So the modulus of the integral also has limit 0 as $\Delta z \to 0$

$\square$.