Research Experiences for Undergraduates

REU 's.

(summer programs).

Solutions to practice problems ↓

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(6) Γ_R = upper half of $|z-2| = R$.

Show $\displaystyle \lim_{R\to\infty} \int_{\Gamma_R} \frac{z^2-3z-6}{z^4+9}\,dz = 0.$

$$\left| \int_{\Gamma_R} \frac{z^2-3z-6}{z^4+9}\,dz \right| \leq ML \quad \text{where}$$

$|f(z)| \leq M$ for all $z \in \Gamma_R$

and $L =$ length of Γ_R

(theorem proved in class).

$$|z^2-3z-6| \leq |z|^2 + 3|z| + 6$$

$z \in \Gamma_R \Rightarrow |z-2| = R$ or $z = 2+Re^{i\theta}$

so $|z| \leq 2+R$

$$\frac{|z|^2+3|z|+6 \leq (2+R)^2+3(2+R)+6.}{|z^4+9| \geq \big||z^4|-9\big|}$$

$$z = 2 + Re^{i\theta} \Rightarrow |z| \geq |R-2|$$

$$|z^4| \geq (R-2)^4$$

$$\big|\, |z^4| - 9 \,\big| \geq |(R-2)^4 - 9|$$

So

$$\left| \frac{z^2 - 3z \mp 6}{z^4 + 9} \right| \leq \frac{(2+R)^2 + 3(2+R) + 6}{(R-2)^4 - 9} = M$$

and

$$\left| \int_{\Gamma_R} \frac{z^2 - 3z \mp 6}{z^4 + 9}\, dz \right| \leq M \cdot \pi R$$

$$\leq \pi R \, \frac{(R+2)^2 + 3(R+2) + 6}{(R-2)^4 - 9}$$

has limit 0 when $R \to \infty$.
divide top & bottom by R^4:

$$\pi \, \frac{\frac{1}{R}\left(1 + \frac{2}{R}\right)^2 + \frac{3}{R^2}\left(1 + \frac{2}{R}\right) + \frac{6}{R^3}}{\left(1 - \frac{2}{R}\right)^4 - \frac{9}{R^4}}$$

as $R \to \infty$ $\begin{cases} \text{numerator} \to 0 \\ \text{denominator} \to 1. \end{cases}$

So lim of the fraction is 0

$$\Rightarrow \lim_{R\to\infty} \int_{\Gamma_R} (\ ) = 0$$

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(7) C_R = circle $|z-1| = R$

$$\int_{C_R} \frac{\text{Log}(z-1)}{z-1}\,dz \qquad z-1 = Re^{i\theta}$$
$$-\pi \le \theta \le \pi$$

$$\text{Log}(z-1) = \ln|z-1| + i\,\text{Arg}(z-1)$$
$$= \ln R + i\theta$$

$$dz = Rie^{i\theta}\,d\theta \cancel{= i(z-1)\,d\theta}$$

$$\int_{C_R} \frac{\text{Log}(z-1)}{z-1}\,dz = \int_{-\pi}^{\pi} \frac{\ln R + i\theta}{Re^{i\theta}}\,Rie^{i\theta}\,d\theta$$

$$= \int_{-\pi}^{\pi} (i\ln R - \theta)\,d\theta$$

$$= i\ln R \int_{-\pi}^{\pi} d\theta - \int_{-\pi}^{\pi} \theta\,d\theta$$

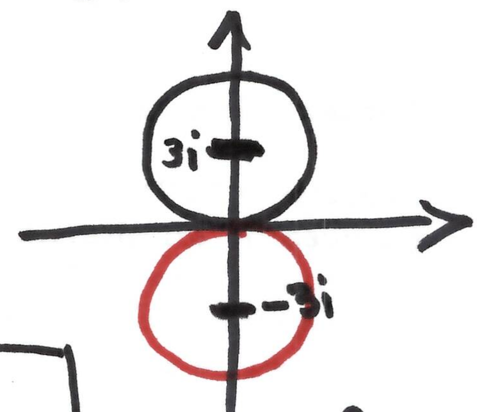
$$= 2\pi i \ln R - \left[\frac{\theta^2}{2}\right]_{-\pi}^{\pi}$$

$$= 2\pi i \ln R - \left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 2\pi i \ln R$$

Depends on R. Because: $\operatorname{Log}(z-1)$ is <u>not</u> analytic along the <u>negative x-axis</u>.

(γ) C = circle of radius 3 centered at $3i$

(a) $$\int_C \frac{e^z}{z^2+9}\,dz$$



$$z^2+9 = (z-3i)(z+3i)$$

$$\int_C \frac{e^z}{z^2+9}\,dz = \int_C \frac{\boxed{e^z/(z+3i)}}{z-3i}\,dz \quad \to f(z)$$

numerator is analytic on C and inside C

$$\Rightarrow \int_C = 2\pi i\, f(z_0) = 2\pi i\, f(3i)$$

$$= 2\pi i \frac{e^{3i}}{3i+3i} = \frac{\pi}{3} e^{3i}$$

(b) C' = circle of radius 3 centered at $-3i$

$$\int_{C'} = \int_{C'} \frac{\boxed{e^z / z-3i}}{z+3i}\, dz \qquad \boxed{e^z / z-3i} \longrightarrow g(z)$$

$g(z)$ is analytic on C' & inside C'

$$\Rightarrow \int_{C'} \frac{g(z)}{z+3i}\, dz = 2\pi i\, g(-3i)$$

$$= 2\pi i \, \frac{e^{-3i}}{-3i-3i} = -\frac{\pi}{3} e^{-3i}$$

other choice: C' = circle of radius 2 centered at 0.

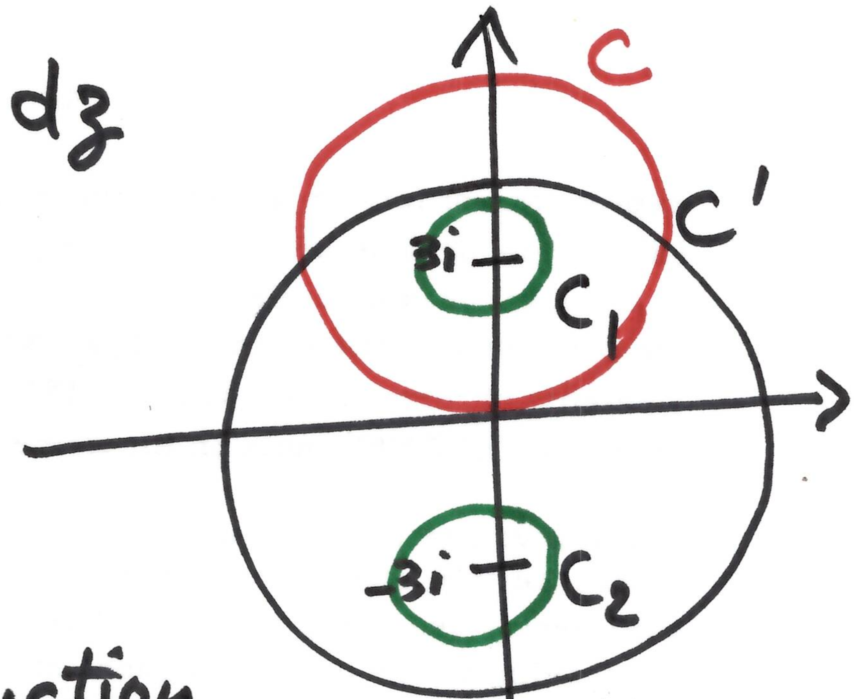
$$\int_{C'} \frac{e^z}{z^2+9}\, dz$$

$\dfrac{e^z}{z^2+9}$ is analytic on C' & inside C'

$$\Longrightarrow \int_{C'} \frac{e^z}{z^2+9}\, dz = 0$$

other choice: C' = circle of radius 4 centered at 0.

$$\int_{C'} \frac{e^{z}}{(z+3i)(z-3i)}\,dz$$



$$\int_{C'} = \int_{C_1} + \int_{C_2}$$

because the function is analytic on C', C_1, C_2 and between C' and $C_1 \cup C_2$.

Similarly $\displaystyle\int_{C_1} = \int_{C} = \frac{\pi}{3} e^{3i}$

$$\int_{C_2} = \int_{\text{circle of radius 3 centered at } -3i}$$

$$= -\frac{\pi}{3} e^{-3i}$$

$$\int_{C'} = \frac{\pi}{3} e^{3i} - \frac{\pi}{3} e^{-3i}$$

$$= \frac{2\pi i}{3} \sin 3$$

(5) (a) $3^{i+1} = 3\, 3^{i}$

$= 3 e^{i \log 3}$

$\log 3 = \ln 3 + i \arg 3$

$= \ln 3 + i 2\pi k \qquad k \in \mathbb{Z}.$

$3^{i+1} = 3 e^{(\ln 3 + i 2\pi k) i}$

$\cancel{= 3 e^{\ln 3} e^{2\pi i k}}$

$= 3 e^{i \ln 3} e^{-2\pi k}.$

$= 3 e^{-2\pi k} \left(\cos(\ln 3) + i \sin(\ln 3) \right)$

(b) $\log(\sin(i+1))$

$= \ln|\sin(i+1)| + i \arg(\sin(i+1))$

$\sin(i+1) = \frac{1}{2i}\left(e^{i(i+1)} - e^{-i(i+1)} \right)$

$= \frac{1}{2i}\left(e^{-1+i} - e^{1-i} \right)$

$= \frac{1}{2i}\left[\frac{e^{i}}{e} - e \cdot e^{-i} \right]$

$$= \frac{1}{2i}\left[\frac{\cos 1 + i\sin 1}{e} - e\left(\cos 1 - i\sin 1\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{e}\left(\sin 1 - i\cos 1\right) + e\left(\sin 1 + i\cos 1\right)\right)$$

$$= \left(\frac{1}{2e}\sin 1 + \frac{e}{2}\sin 1\right)$$

$$+ i\left(\frac{e}{2}\cos 1 - \frac{1}{2e}\cos 1\right)$$

$$= \left(\frac{1}{2e} + \frac{e}{2}\right)\sin 1 + i\left(\frac{e}{2} - \frac{1}{2e}\right)\cos 1$$

$$r = \left|\sin(i+1)\right| = \sqrt{\left(\quad\right)^2 + \left(\quad\right)^2}$$

the argument is in the first quadrant so we can take

$$\theta = \arctan\left(\frac{\operatorname{Im}(\sin(i+1))}{\operatorname{Re}(\sin(i+1))}\right) = \operatorname{Arg}(\sin(i+1))$$

$$\log(\sin(i+1)) = \ln r + i(\theta + 2k\pi).$$

$$k \in \mathbb{Z}.$$

(4) $f(z) = f(x+iy) = x^2 + iy^2$

(a) f continuous everywhere because Ref & Imf are continuous everywhere

(b) Cauchy-Riemann: if f is differentiable at $z = x+iy$ ($f = u+iv$), then

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$u_x = 2x \qquad v_y = 2y$$

$$u_y = 0 \qquad v_x = 0$$

$$u_x = v_y \implies 2x = 2y. \Rightarrow x = y.$$

If $u_x = v_y$ & $u_y = -v_x$ & u_x, u_y, v_x, v_y are continuous at z_0, then f is differentiable at z_0.

The partials are continuous everywhere, so, according to Cauchy-Riemann, f is differentiable when $x = y$.

(c) Nowhere analytic:

$\forall$ $z \in$ line $(x=y)$ any neighborhood of z contains points where f is <u>not</u> differentiable.