Another consequence of the Cauchy integral formulas:

<u>Corollary</u>: Suppose f is analytic on and inside ~~the~~ circle C of radius R with center z_0 and that ~~[illegible]~~ s.t. $|f(z)| \leq M \quad \forall z \in C$.

Then $$|f^{(n)}(z_0)| \leq \frac{n!\, M}{R^n} \quad \forall n \geq 0$$

<u>Proof</u>: $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

$$\left| \frac{f(z)}{(z-z_0)^{n+1}} \right| \leq \frac{M}{R^{n+1}}$$

so, by theorem ~~a~~ in Section 47:

$$\left| \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \right| \leq \frac{M}{R^{n+1}} (\text{length of } C)$$

So $$|f^{(n)}(z_0)| \leq \frac{M}{R^{n+1}} \cdot 2\pi R \cdot \frac{n!}{2\pi} = \frac{n!\, M}{R^n}$$

$\square$

Liouville's theorem: ~~A~~ A bounded entire function is constant.

Proof: $\forall\, z_0 \in \mathbb{C}$ and $\forall$ circles of center z_0, by the corollary above:

$$|f'(z_0)| \leq \frac{M}{R} \quad \text{where}$$

$R =$ radius of circle and M is a ~~the~~ bound for $|f(z)|$ on all of $\mathbb{C}$

We can take R to be arbitrarily large $\Rightarrow |f'(z_0)| = 0$

$\Rightarrow f$ is constant. $\square$.

Consequence:

The fundamental theorem of algebra:

Any polynomial with complex coefficients and degree ≥ 1 has at least one complex root.

Proof: By contradiction, Assume $P(z)$ is a polynomial with no roots. Then $\frac{1}{P(z)}$ is analytic where $P(z) \neq 0$, meaning everywhere. So $\frac{1}{P(z)}$ is entire.

On 01/12/2017 we proved that $\exists\ R > 0$ s.t. if $|z| \geq R$, then $|P(z)| \geq \frac{|a_n|}{2} R^n$ where $n = \text{degree}(P)$ and a_n is the leading coefficient of P, meaning $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots$

So $\left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n| R^n}$, if $|z| \geq R$

The ~~circle~~ disc $|z| \leq R$ is compact and $\left|\frac{1}{P(z)}\right|$ is continuous on the disc, so

it is bounded. So $\exists\ m$ s.t.
if $|z| \leq R$, then $\left|\frac{1}{P(z)}\right| \leq m$

Take $M \geqslant \operatorname{Max}\left\{m, \frac{2}{|a_n| R^n}\right\}$.

then $\left|\frac{1}{P(z)}\right| \leq M \quad \forall z \in \mathbb{C}$.

So $\frac{1}{P(z)}$ is entire and bounded, hence constant by Liouville.

So $P(z) =$ constant and has degree 0. $\square$

<u>Consequences:</u> $P(z)$, $n =$ degree $\geqslant 1$

$\exists\ \alpha_1$ root of $P(z)$.

so $P(\alpha_1) = 0$

or $(z - \alpha_1) \mid P(z)$.

or $\exists$ polynomial $Q(z)$ of degree $n-1$

s.t. $P(z) = Q(z)(z - \alpha_1)$

if $n \geqslant 2$, deg $Q \geqslant 1$ and Q has a root α_2.

then $P(z) = R(z)(z-\alpha_1)(z-\alpha_2)$

$\deg R(z) = n-2$

continue to get:

$$P(z) = a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$$

the number of times a given α is equal to one of the α_i is the ~~mult~~ multiplicity of α. So

$$P(z) = a_n(z-\beta_1)^{m_1}(z-\beta_2)^{m_2}\cdots(z-\beta_r)^{m_r}$$

m_i = multiplicity of β_i

$\beta_i \neq \beta_j \quad \forall\ i \neq j$

and $m_1 + \cdots + m_r = n$

We say the number of roots, counting multiplicities, is n = degree.

If P has real coefficients and $P(\alpha) = 0$, then $P(z) = a_n z^n + \cdots + a_0$

$a_i \in \mathbb{R}$

$$0 = \overline{P(\alpha)} = \overline{a_n\alpha^n + \cdots + a_0}$$

$$= \overline{a_n}\,\overline{\alpha}^n + \cdots + \overline{a_0} = a_n\overline{\alpha}^n + \cdots + a_0$$

$$= P(\overline{\alpha})$$

$$(z-\alpha)(z-\bar{\alpha}) \mid P(z), \text{ if } \alpha \neq \bar{\alpha}$$

$$\| $$

$$z^2-(\alpha+\bar{\alpha})z+\alpha\bar{\alpha} = z^2-(2\operatorname{Re}\alpha)z+|\alpha|^2$$

So $P(x) = a_n(x-r_1)\cdots(x-r_m)$

$$\cdot(x-\alpha_1)(x-\bar{\alpha}_1)\cdots(x-\alpha_\ell)(x-\bar{\alpha}_\ell)$$

α_i imaginary $(\operatorname{Im}\alpha_i \neq 0)$

r_i real $(\operatorname{Im} r_i = 0)$

So $P(x) =$ product of linear and quadratic factors with real coefficients.

$$(x-\alpha_i)(x-\bar{\alpha}_i) = x^2-(2\operatorname{Re}\alpha_i)x+|\alpha_i|^2$$

is quadratic with real coeff.

Lemma: Suppose f analytic on an ε-neighborhood of $z_0 \in \mathbb{C}$ and on the boundary of the ε-neighborhood.

If $|f(z)| \leq |f(z_0)|$ for all $z \in$ neighborhood, then f is constant on the neighborhood. $(= f(z_0))$

Proof: By Cauchy:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

C = circle of radius ε, center z_0

parametrize the circle:

$$z = z_0 + \varepsilon e^{i\theta} \qquad 0 \le \theta \le 2\pi.$$

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\varepsilon e^{i\theta})\,d\theta.$$

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\varepsilon e^{i\theta})|\,d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta.$$

$$= \frac{1}{2\pi}|f(z_0)|\cdot\int_0^{2\pi} d\theta = |f(z_0)|$$

So $$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\varepsilon e^{i\theta})|\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta.$$

So $$\int_0^{2\pi} \left(|f(z_0)| - |f(z_0+\varepsilon e^{i\theta})|\right) d\theta = 0$$

on circle $|f(z_0)| \geq |f(z)|$

So $\boxed{|f(z_0)| - |f(z)| \geq 0}$

since $\int_0^{2\pi} (|f(z_0)| - |f(z)|)\, d\theta = 0$.

we must have $|f(z_0)| - |f(z)| = 0$

on circle.

So $|f(z)| = |f(z_0)|$ $\forall z \in C$

True $\forall$ circle of radius small enough with center z_0.

So $|f(z)| = |f(z_0)|$ on some disc with center z_0.

We showed a while ago that this implies $f(z)$ is constant on the disc. So $f(z) = f(z_0)$ on some disc with center z_0. $\square$

<u>Theorem</u> (Maximum Modulus Principle): f analytic on a domain D. Then $|f(z)|$ has <u>no</u> maximum on D, if f is NOT constant.

Corollary: If f is analytic on the interior of a compact subset $K \subset \mathbb{C}$ and continuous on the boundary of K, then f reaches its maximum on the boundary of K and not on the interior.

Proof of theorem: Prove the contrapositive:

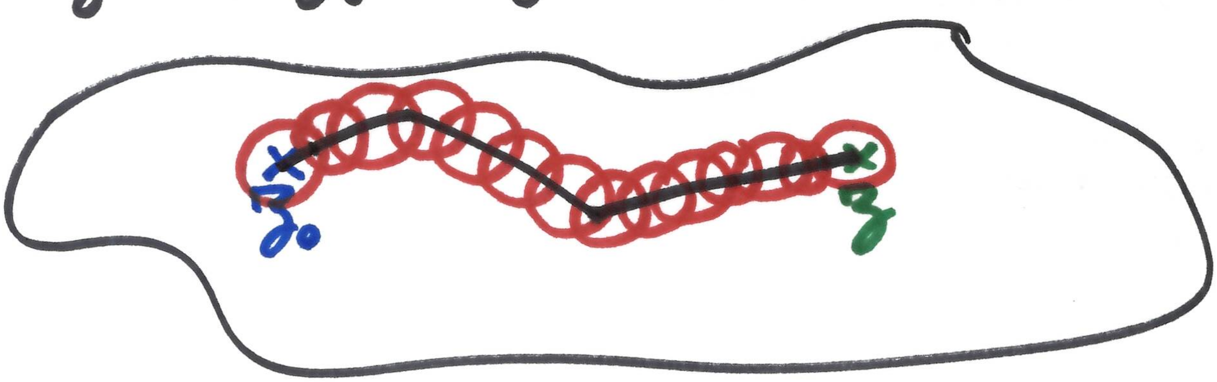
If $|f|$ has a maximum on D, then f is constant on D.

Suppose $|f|$ has a maximum, say M, reached at, say $z_0 \in D$.

So $|f(z_0)| = M$ and

$$\forall z \in D \quad |f(z)| \leq M$$

$z \in D$ arbitrary. We can join z to z_0 by a (polygonal) path.



Cover path with circles $C_0, \dots, C_n$
C_0 with center z_0. C_i with radius ε_i and center z_i, ε_i small enough so that the entire disc is in D.

On C_0: $|f(z)| \leq |f(z_0)|$
$\forall$ $z \in$ disc inside C_0.

By lemma: $|f(z)| = |f(z_0)|$ on the disc. and $f(z) = f(z_0)$

Increase number of circles if necessary so that the center of the next circle is inside the previous one.

Then for C_1: $|f(z)| \leq |f(z_1)|$ on the disc inside C_1

$$\Rightarrow \quad f(z) = f(z_1) = f(z_0)$$

for all $z \in$ disc inside C_1.

Continue to the last $z_n = z$ to get $f(z) = f(z_0)$ $\forall$ $z \in D$.

$\square$