Power series expansions of analytic functions.

$\{a_n\}_{n \in \mathbb{N}}$ $\qquad a_n \in \mathbb{C}$

We say that the limit of $\{a_n\}$ is L as $n \to +\infty$ if $\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$ $\quad |a_n - L| < \varepsilon$.

The series of general term a_n is denoted $\sum a_n$. The partial sums ~~We say the~~ of the series is $S_n = \sum_{k=0}^{n} a_n$

this is a sequence.

We say the sum of the series is L if L is the limit of $\{S_n\}$.

We write $\sum_{n=0}^{\infty} a_n = L$

We can write $a_n = x_n + i y_n$

$x_n, y_n \in \mathbb{R}$.

if $L = a + bi$

then $\lim_{n\to\infty} a_n = L \iff \begin{cases} \lim_{n\to\infty} x_n = a \\ \& \lim_{n\to\infty} y_n = b. \end{cases}$

We say $\sum a_n$ is absolutely convergent, if $\sum |a_n|$ is convergent.

Theorem: If $\sum a_n$ is absolutely convergent, then it is convergent.

Prop.: If $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ (converse not true).

$\sum \frac{1}{n^r}$ $0 < r \leq 1$ diverges

Important example: geometric series.

$z \in \mathbb{C}$ $1, z, z^2, z^3.$

$$\sum z^n$$

$$S_n = \sum_{k=0}^{n} z^k = 1 + z + \cdots + z^n.$$

$$S_n z = z + z^2 + \cdots + z^{n+1}$$

$$S_n(1-z) = 1 - z^{n+1}$$

If $z=1$ series diverges. (general term $=1$)

If $z \neq 1$ $S_n = \dfrac{1-z^{n+1}}{1-z}.$

If $|z|<1$ then $\lim_{n\to\infty} |z|^{n+1} = 0$

and $\lim_{n\to\infty} S_n = \dfrac{1}{1-z}$

So $\sum z^n$ converges to $\dfrac{1}{1-z}.$

$\sum z^n$ is absolutely convergent:

$|z^n| = |z|^n$ $S_n = \dfrac{1-|z|^{n+1}}{1-|z|}$

and $\sum_{n=0}^{\infty} |z|^n = \dfrac{1}{1-|z|}$

If a series is absolutely convergent, any other series obtained by changing the order of the terms is also absolutely convergent and has the same sum.

If the series is not absolutely convergent, the convergence and the value of the sum (if it exists) depend on the ordering of the terms.

($\sum a_n$, ordering of the a_n)

reordering the a_n changes the S_n:

$a_0, a_1, a_2, a_3, \cdots$

$S_0 = a_0,\ S_1 = a_0 + a_1, \cdots$

$b_0 = a_0,\ b_1 = a_3,\ b_2 = a_4,\ b_3 = a_1, \cdots$

$S_0 = b_0 = a_0,\ S_1 = b_0 + b_1 = a_0 + a_3$

E.g. $\sum \frac{(-1)^n}{n}$ $\quad -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

Can rearrange the terms to get any real number as the sum, can also get $+\infty$ or $-\infty$.

Taylor's theorem: Suppose f is analytic on a disc $|z - z_0| < R$ then $\forall\ z$ in the open disc:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ where } a_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof: We can assume $z_0 = 0$

e.g. $w = z - z_0$

$$f(z) = f(w + z_0) = g(w).$$

$$f^{(n)}(z_0) = g^{(n)}(0) \quad \forall n$$

by chain rule.

From now on assume $z_0 = 0$.

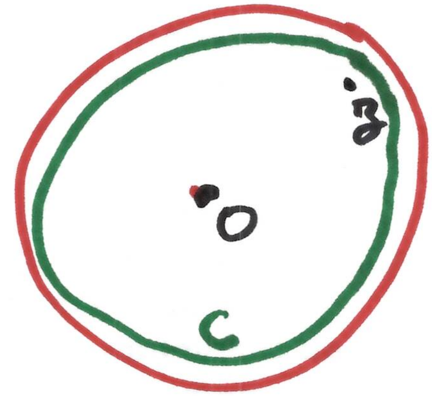
disc $|z| < R$.

Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z}\, ds.$$

C = circle of radius $r_0 < R$, center 0

s.t. z is inside C.



$$\frac{1}{s - z} = \frac{1}{s} \, \frac{1}{1 - \frac{z}{s}}$$

$$0 \le \left| \frac{z}{s} \right| = \frac{r}{r_0} < 1 \qquad r = |z|$$

$$\frac{1}{1-\frac{z}{s}} = \sum_{n=0}^{\infty}\left(\frac{z}{s}\right)^n$$

$$= \sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \sum_{n=N}^{\infty}\left(\frac{z}{s}\right)^n$$

$$= \sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \left(\frac{z}{s}\right)^N \sum_{n=0}^{\infty}\left(\frac{z}{s}\right)^n$$

$$= \sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \left(\frac{z}{s}\right)^N \frac{1}{1-\frac{z}{s}}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s\left(1-\frac{z}{s}\right)}\,ds$$

$$= \frac{1}{2\pi i}\left[\int_C \frac{1}{s}\sum_{n=0}^{N-1} f(s)\left(\frac{z}{s}\right)^n ds + \int_C \frac{1}{s} f(s)\left(\frac{z}{s}\right)^N \frac{ds}{1-\frac{z}{s}}\right]$$

$$= \frac{1}{2\pi i}\left(\sum_{n=0}^{N-1}\int_C \frac{f(s)}{s^{n+1}}\,ds\right) z^n + \left(\frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s^N(s-z)}\right) z^N$$

$$= \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \frac{1}{2\pi i}\left(\int_C \frac{f(s)\,ds}{s^N(s-z)}\right) z^N.$$

$$= \sum_{n=0}^{N-1} a_n z^n + \frac{1}{2\pi i}\left(\int_C \frac{f(s)\,ds}{s^N(s-z)}\right) z^N.$$

Need to prove $\displaystyle\lim_{N\to\infty} \int_C \frac{f(s)\,ds}{s^N(s-z)} z^N = 0$

$$\left|\left(\int_C \frac{f(s)\,ds}{s^N(s-z)}\right) z^N\right| \leq M \text{ length}(C) \cdot |z|^N$$

if $M \geq \left|\frac{f(s)}{s^N(s-z)}\right|$ for all $s \in C$

f is continuous on C, hence bounded $\quad |s| = r_0$ on C

Choose a bound B for $|f|$ on C.

then $\displaystyle\left|\frac{f(s)}{s^N(s-z)}\right| \leq \frac{B}{r_0^N(r_0 - r)}$

<u>because</u>:

$$|s-z| \geq |s| - |z| = r_0 - r > 0$$

So

$$\left|\int_C \frac{f(s)\,ds}{s^N(s-z)} z^N\right| \leq \frac{B}{r_0^N(r_0-r)} r^N$$

$$= \frac{B}{r_0-r}\left(\frac{r}{r_0}\right)^N.$$

$\Rightarrow$ the limit is 0 as $N \to \infty$
because $0 < \frac{r}{r_0} < 1$ $\square$

Examples: $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$

expansion centered at 0.

similarly for $\sin z$, $\cos z$, etc.

other example: $\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2} + \cdots$

a Laurent series expansion.
the function has a singularity

Laurent's theorem: Suppose f is analytic on an annulus

$$0 \leq R_1 < |z - z_0| < R_2$$

suppose C is a simple closed contour in the annulus and encloses z_0, (C oriented counterclockwise).

then $\forall z$ in the annulus

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

where
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \int_C f(z)(z-z_0)^{n-1}\,dz.$$

<u>Shorthand</u>: $f(z) = \sum_{-\infty}^{+\infty} c_n (z-z_0)^n$

where $c_n = a_n$ if $n \geq 0$
$c_n = b_{-n}$ if $n < 0$
