Assume $z_0 = 0$ like before

Proof of Laurent's theorem: $z \in$ annulus

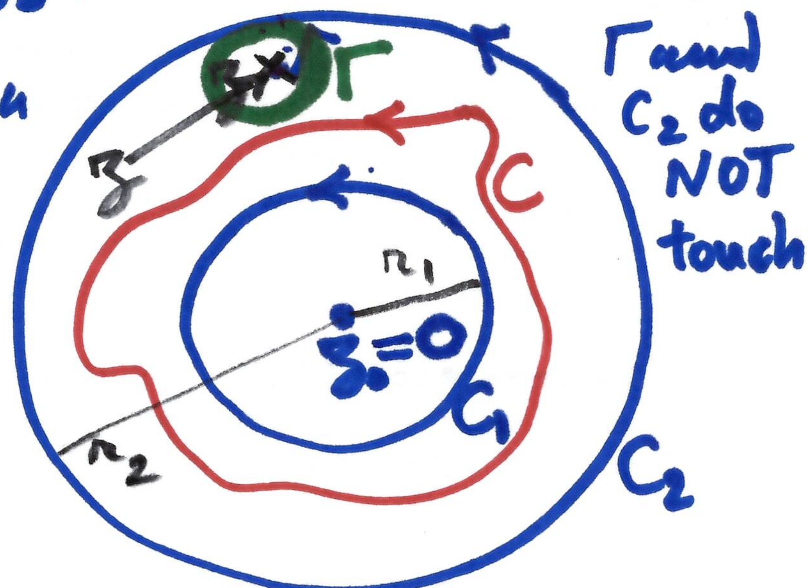
Choose $r_1 > R_1$ and $r_2 < R_2$

s.t. $R_1 < r_1 < |z - z_0| < r_2 < R_2 \le +\infty$

and C is between the circles

$C_1 : |z - z_0| = r_1$

and $C_2 : |z - z_0| = r_2$



also choose $\gamma > 0$ s.t. the circle Γ of radius γ and center z is between C_1 and C_2 (and C)

Cauchy integral formula:

$$2\pi i f(z) = \int_\Gamma \frac{f(s)\,ds}{s - z}.$$

$\dfrac{f(s)}{s - z}$ is analytic on C_1, C_2, Γ and between them, so:

$$\int_{C_2} \frac{f(s)}{s - z} ds = \int_{C_1} \frac{f(s)\,ds}{s - z} + \int_\Gamma \frac{f(s)\,ds}{s - z}$$

So

$$2\pi i f(z) = \int_{\Gamma} \frac{f(s)\,ds}{s-z}$$

$$= \int_{C_2} \frac{f(s)\,ds}{s-z} + \int_{C_1} \frac{f(s)\,ds}{z-s}$$

$$\frac{1}{s-z} = \frac{1}{s} \cdot \frac{1}{1-\frac{z}{s}} \quad \text{on } C_2 \quad \left|\frac{z}{s}\right| < 1$$

$$= \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{z}{s}\right)^n$$

$$\frac{1}{z-s} = \frac{1}{z} \frac{1}{1-\frac{s}{z}} \quad s \in C_1 \quad \left|\frac{s}{z}\right| < 1$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{s}{z}\right)^n$$

As in the proof of Taylor's theorem:

$$\frac{1}{s-z} = \frac{1}{s}\left(\sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \left(\frac{z}{s}\right)^N \sum_{n=0}^{\infty} \left(\frac{z}{s}\right)^n \right)$$

$$= \frac{1}{s}\left(\sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \left(\frac{z}{s}\right)^N \frac{1}{1-\frac{z}{s}} \right)$$

and

$$\frac{1}{z-s} = \frac{1}{z}\left(\sum_{n=0}^{N-1} \left(\frac{s}{z}\right)^n + \left(\frac{s}{z}\right)^N \frac{1}{1-\frac{s}{z}} \right)$$

and:

$$2\pi i f(z) = \int_{C_2} \frac{f(s)}{s} \sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n ds$$

$$+ \int_{C_1} \frac{f(s)}{z} \sum_{n=0}^{N-1} \left(\frac{s}{z}\right)^n ds.$$

$$+ \int_{C_2} \frac{f(s)}{s}\left(\frac{z}{s}\right)^N \frac{ds}{1-\frac{z}{s}}$$

$$+ \int_{C_1} \frac{f(s)}{z}\left(\frac{s}{z}\right)^N \frac{ds}{1-\frac{s}{z}}$$

$$= \left\{ \sum_{n=0}^{N-1} \left[z^n \int_{C_2} \frac{f(s)}{s^{n+1}} ds + z^{-n-1} \int_{C_1} f(s) s^n ds \right] \right.$$

$$+ z^N \int_{C_2} \frac{f(s)\,ds}{s^N(s-z)} + z^{-N} \int_{C_1} \frac{s^N f(s)\,ds}{z-s}$$

because f is analytic on C, C_1, C_2 and between any two of them

$$\int_C = \int_{C_1} = \int_{C_2} \quad \text{above.}$$

So $$\int_{C_2} \frac{f(s)}{s^{n+1}}\,ds = 2\pi i\, a_n$$

and $$2\pi i\, b_{n+1} = \int_{C_1} f(s)\, s^n\, ds$$

To finish the proof, we need to show

$$\lim_{N\to\infty} z^N \int_{C_2} \frac{f(s)\,ds}{s^N(s-z)} = 0$$

and separately

$$\lim_{N\to\infty} z^{-N} \int_{C_1} \frac{s^N f(s)\,ds}{z-s} = 0$$

Proof for the first one:

$$\left|\int_{C_2} \frac{z^N f(s)\,ds}{s^N(s-z)}\right| \leq \text{length of } C_2 \cdot \text{max. modulus of function in integral}$$

$$\left|\frac{z^N}{s^N}\frac{f(s)}{s-z}\right| \leq \left(\frac{|z|}{r_2}\right)^N \frac{|f(s)|}{r_2-|z|}$$

<u>because</u>: $|s-z| \geq |s|-|z| = r_2 - |z| = \text{constant}$

Choose some M s.t. $M \geq |f(s)|$ for all $s \in C_2$.

so $$\left|\frac{z^N}{s^N}\frac{f(s)}{s-z}\right| \leq \frac{M}{r_2-|z|}\left(\frac{|z|}{r_2}\right)^N.$$

$0 < \frac{|z|}{r_2} < 1$ so limit as $N \to \infty$ is 0.

<u>Some useful properties</u>:

<u>Uniqueness</u>: If f is analytic on a disc (or annulus) and we have a power series or ~~the~~ ~~su~~ (a Laurent series) which converges to

f on the disc (or annulus), then the series is the Taylor series (or the Laurent series).

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e.g.: $f(z) = \dfrac{1}{z(z-1)}$ center 0.

$$= \frac{-1}{z}\left(\frac{1}{1-z}\right) = -\frac{1}{z}\sum_{n=0}^{\infty} z^n$$

converges to f when $|z| < 1$

annulus: $\underset{R_1}{\overset{\shortparallel}{0}} < |z| < \underset{R_2}{\overset{\shortparallel}{1}}$

$$f(z) = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$$

So $b_n = 0$ if $n \geq 2$, $b_1 = -1$

$a_n = -1$ for all n.

So $-1 = \dfrac{1}{2\pi i}\displaystyle\int_C f(z)\,dz = b_1$

<u>Properties of power series</u> (non negative powers of z only)

The largest R s.t. a power series converges on the disc $|z-z_0|<R$ is called the <u>radius of convergence</u>.

The <u>circle of convergence</u> is the circle $|z-z_0|=R$.

The series <u>converges absolutely</u> on the open disc $|z-z_0|<R$.

It may or may not converge on the circle of convergence.

The sum of a convergent series is <u>analytic</u> on the open disc $|z-z_0|<R$

It is legal to differentiate a convergent series term by term inside the open disc $|z-z_0|<R$, and also to integrate term by term on contours inside $|z-z_0|<R$.

We can multiply and divide series (like polynomials).

Examples: ① $e^{1/z}$ analytic for $z \neq 0$.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n \quad \text{center} = 0$$

$$= \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{z^{-n}}{n!}$$

$a_0 = 1 \quad a_n = 0 \quad n \geq 1$

$b_n = \frac{1}{n!}$ for $n \geq 1$

this gives all the contour integrals around 0. e.g.

$$a_0 = 1 = 2\pi i \int_C \frac{e^{1/z}\,dz}{z}$$

$$a \quad b_1 = 1 = 2\pi i \int_C e^{1/z}\,dz$$

② $\frac{1}{\sin z}$ center 0.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\sin z = z\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\cdots\right)$$

$$\frac{1}{\sin z}=\frac{1}{z}\,\frac{1}{1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\cdots}$$

$$1+\frac{z^2}{3!}+\left(\frac{1}{3!3!}-\frac{1}{5!}\right)z^4+\cdots$$

$$1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\cdots\ \Big)\ 1+0\cdot z+0z^2+\cdots$$

$$-\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\cdots\right)$$

$$\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}-\cdots$$

$$-\left(\frac{z^2}{3!}-\frac{z^4}{(3!)^2}+\frac{z^6}{3!5!}-\right.$$

$$\left(\frac{1}{(3!)^2}-\frac{1}{5!}\right)z^4+\left(\frac{1}{7!}-\frac{1}{3!5!}\right)z^6+\cdots$$

$$\frac{1}{\sin z}=\frac{1}{z}+\frac{z}{3!}+\left(\frac{1}{(3!)^2}-\frac{1}{5!}\right)z^3+\cdots$$

③ $\frac{1}{z} e^z \cos z$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = 1 - \frac{1}{2} z^2 + \frac{1}{24} z^4 - \cdots$$

$$\frac{1}{z} e^z \cos z = \frac{1}{z}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right)\left(1 - \frac{1}{2} z^2 + \frac{1}{24} z^4 - \cdots\right)$$

$$= \frac{1}{z}\left(1 + z + 0z^2 + -\frac{1}{3} z^3 + \cdots\right)$$

$$= \frac{1}{z} + 1 - \frac{1}{3} z^2 + \cdots$$

④ $\operatorname{Log}(1+z) = f(z)$

$$f'(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n$$

$\boxed{|z| < 1}$ center 0.

So $\operatorname{Log}(z+1) = C + \sum_{n=0}^{\infty} (-1)^n \frac{(z)^{n+1}}{n+1}$

$z = 0 \quad \operatorname{Log} 1 = 0 = C$

$$\Rightarrow \operatorname{Log}(z+1) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1}$$