Solutions to practice problems:

(2) $$\int_0^{2\pi} e^{iN\theta} F(z_0 + Re^{i\theta})\, d\theta$$

(a) $z = z_0 + Re^{i\theta}$ $\quad dz = Rie^{i\theta} d\theta$

$= i(z - z_0)\, d\theta$

$$\int_C (z - z_0)^N \frac{1}{R^N} F(z) \frac{1}{i(z - z_0)}\, dz$$

$$= \frac{1}{iR^N} \int_C F(z)(z - z_0)^{N-1}\, dz$$

(b) If $N - 1 \geq 0$, then $F(z)(z - z_0)^{N-1}$ is analytic on and inside C.

Cauchy-Goursat $\Rightarrow \int_C F(z)(z - z_0)^{N-1} dz = 0$

If $N - 1 < 0$ $\quad (N < 1)$

then $$= \frac{1}{iR^N} \int_C \frac{F(z)}{(z - z_0)^{-N+1}}\, dz.$$

$\underbrace{-N}+1 > 0 \qquad -N \geq 0$

By the Cauchy integral formulas:

$$= \frac{1}{iR^N} 2\pi i \frac{F^{(-N)}(z_0)}{(-N)!} = \frac{2\pi F^{(-N)}(z_0)}{R^N (-N)!}$$

$$= \frac{2\pi}{(-N)!} R^{-N} F^{(-N)}(z_0)$$

(3)(a) $\text{Log}(2z+1)$ centered at $z_0 = i$.

$$\frac{d}{dz} \text{Log}(2z+1) = \frac{2}{2z+1}$$

$$\frac{2}{2z+1} = \frac{2}{2(z-i)+2i+1}$$

$$= \frac{2}{2i+1} \frac{1}{1+\frac{2}{2i+1}(z-i)}$$

convergent geometric series provided

$$\left| \frac{2}{2i+1}(z-i) \right| < 1$$

assume this

$$\frac{2}{2z+1} = \frac{2}{2i+1} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{2i+1}\right)^n (z-i)^n (-1)^n$$

Log(2z+1) is the anti-derivative taking the value Log(2i+1) at z=i

$$\operatorname{Log}(2z+1) = \frac{2}{2i+1} \sum_{n=0}^{\infty} \left(\frac{-2}{2i+1}\right)^n \frac{(z-i)^{n+1}}{n+1} + C$$

$$\operatorname{Log}(2i+1) = C$$

So $$\operatorname{Log}(2z+1) = \operatorname{Log}(2i+1) + \frac{2}{2i+1} \sum_{n=0}^{\infty} \left(\frac{-2}{2i+1}\right)^n \frac{(z-i)^{n+1}}{n+1}$$

$$\operatorname{Log}(2i+1) = \ln\sqrt{5} + i\operatorname{Arg}(2i+1)$$

(b) The series converges when

$$\left|\frac{2}{2i+1}(z-i)\right| < 1$$

diverges when $\left|\frac{2}{2i+1}(z-i)\right| > 1$

So the largest Radius for which it converges is $|z-i| < \left|\frac{2i+1}{2}\right|$

$$R = \left|\frac{2i+1}{2}\right| = \frac{\sqrt{5}}{2}.$$

(4) $f(z) = e^{(i-\sqrt{2})\mathrm{Log}\, z}$

$$\int_C f(z)\,dz = \int_C e^{(i-\sqrt{2})\mathrm{Log}\, z}\,dz$$

parametrize C: $z = Re^{i\theta}$

$\theta \in [-\pi, \pi]$

$\mathrm{Log}\, z = \ln R + i\theta$

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} iR\, e^{(i-\sqrt{2})(\ln R + i\theta)}\, e^{i\theta}\,d\theta$$

$$= iR \int_{\pi}^{\pi} e^{i\ln R}\, e^{-\theta}\, e^{-\sqrt{2}\ln R}\, e^{-i\sqrt{2}\,\theta}\, e^{i\theta}\,d\theta$$

$$= iR^{1-\sqrt{2}}\, e^{i\ln R} \int_{-\pi}^{\pi} e^{\theta(-1+i(1-\sqrt{2}))}\,d\theta$$

$$= iR^{1-\sqrt{2}}\, e^{i\ln R} \left[\frac{e^{\theta(-1+i(1-\sqrt{2}))}}{-1+i(1-\sqrt{2})} \right]_{-\pi}^{\pi}$$

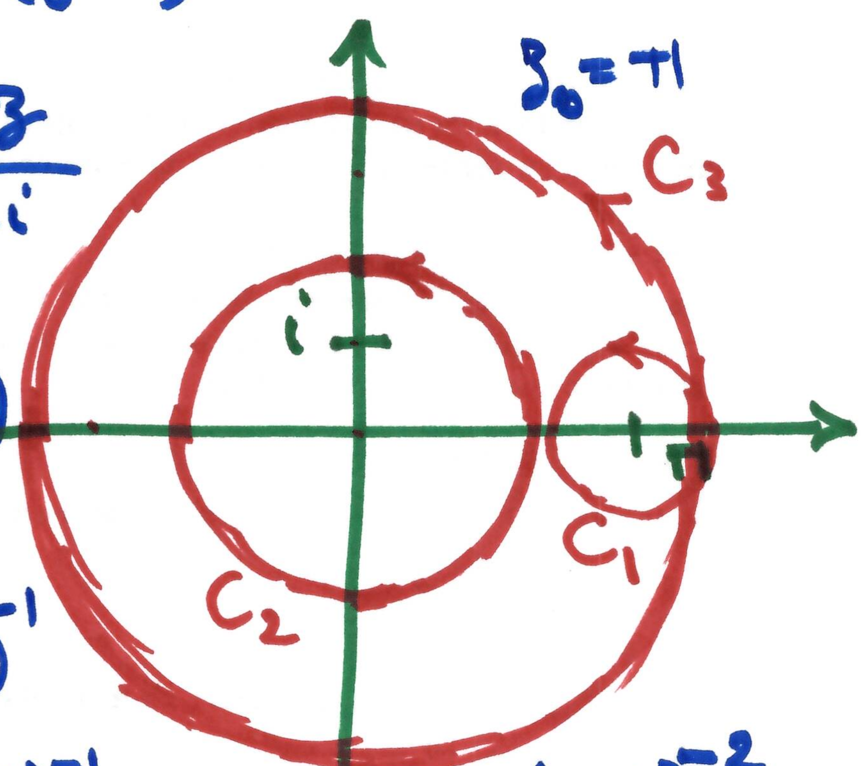
$$\stackrel{*}{=} \frac{i R^{1-\sqrt{2}} e^{i\ln R}}{-1+i(1-\sqrt{2})}\left(e^{-\pi+i\pi(1-\sqrt{2})} - e^{\pi-i\pi(1-\sqrt{2})}\right)$$

$$= \frac{R^{1-\sqrt{2}} e^{i\ln R}}{i+(1-\sqrt{2})}\left(e^{\pi+i\pi\sqrt{2}} - e^{-\pi-i\pi\sqrt{2}}\right)$$

(5) $$\int_C \frac{\cos z}{(z-\pi)^2(z-i)}\,dz$$

(a) $f(z) := \dfrac{\cos z}{z-i}$

$$\int_{C_1} \frac{f(z)\,dz}{(z-\pi)^2} = 2\pi i f'(\pi)$$



$$f(z) = (\cos z)(z-i)^{-1}$$

$$f'(z) = -\sin z\,(z-i)^{-1} + \cos z\,(-1)(z-i)^{-2}$$

$$f'(\pi) = 0 + (-1)(-1)(\pi-i)^{-2}$$

$$= \frac{1}{(\pi-i)^2} = \frac{1}{\pi^2-1-2\pi i}$$

So $\displaystyle\int_{C_1} = 2\pi i \, \frac{1}{\pi^2 - 1 - 2\pi i}$

(b) $\displaystyle\int_{\text{medium circle}}$ define $g(z) = \dfrac{\cos z}{(z-\pi)^2}$

$$\int_{C_2} \frac{g(z)\,dz}{z-i} = 2\pi i \, g(i) \qquad z_0 = i$$

$$= 2\pi i \, \frac{\cos i}{(i-\pi)^2}$$

$$= \frac{2\pi i}{\pi^2 - 1 - 2\pi i} \, \frac{e^{i \cdot i} + e^{-i \cdot i}}{2}$$

$$= \frac{\pi i}{\pi^2 - 1 - 2\pi i} \left(e^{-1} + e\right)$$

(c) $\displaystyle\int_{C_3} = \int_{\substack{C_4 \\ \text{circle}}}$ of radius 5 around 0.
because function is analytic on both circles and between them.

$\displaystyle\int_{C_4} = \int_{C_1} + \int_{C_2}$ because function is analytic on C_1, C_2, C_4 and between them.

So $\displaystyle\int_{C_3} \frac{\cos z \, dz}{(z-\pi i)^2 (z-i)} =$

$$\frac{2\pi i}{\pi^2 \mp 1 - 2\pi i} + \frac{\pi i}{\pi^2 - 1 - 2\pi i}(e^{-1} + e)$$

(6) $f(z) = f(x,y) = u(x,y) + i v(x,y)$

$a, b, c \in \mathbb{R}$, nonzero

s.t. $a\, u(x,y) + b\, v(x,y) = c$

$\forall\, x, y \in \mathbb{R}$

Show f is constant.

$f'(z) = u_x + i v_x$ by Cauchy-Riemann.

and $u_x = v_y$ & $u_y = -v_x$ $\forall\, x, y$

So $u_x = v_y$ $\qquad a u + b v = c$

$\qquad a u_x + b v_x = 0$

So $u_x = -\frac{b}{a} v_x = \frac{b}{a} u_y$

$\qquad a u_y + b v_y = 0$

so $u_y = -\frac{b}{a} v_y$

so $u_x = \frac{b}{a} u_y = -\frac{b^2}{a^2} v_y = -\frac{b^2}{a^2} u_x$

So $\left(1+\frac{b^2}{a^2}\right)u_x = 0$ and $1+\frac{b^2}{a^2} \neq 0$ because a and b are real.

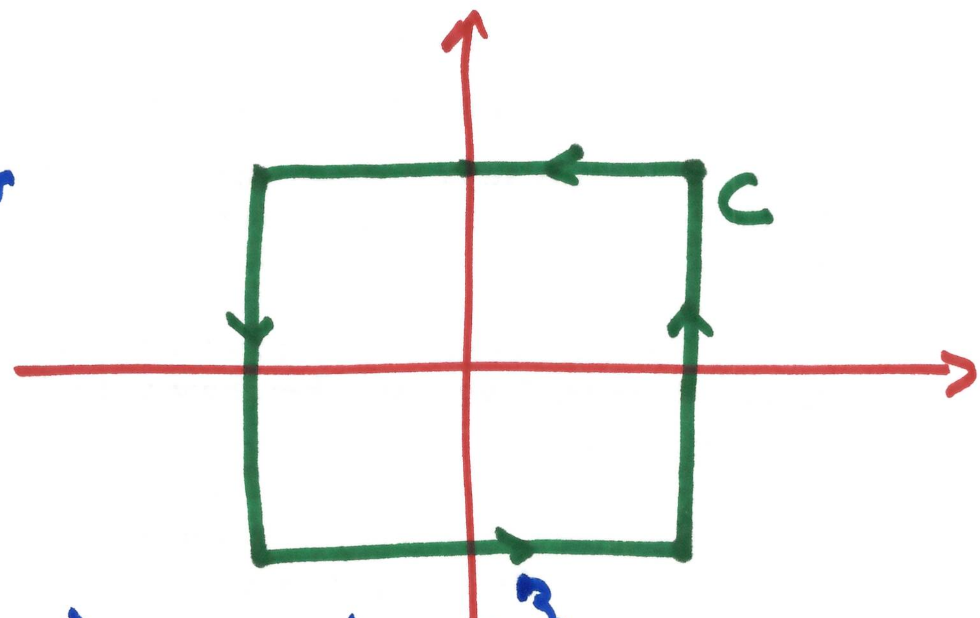
So $u_x = 0 \quad \forall\, x, y$

$= v_y = u_y = v_x$

So $f'(z) = 0 \quad \forall\, z$

So f is constant.

(1) (a) $\displaystyle\int_C \frac{e^z}{z+2i}\,dz$



$|I| \leq (\text{length of } C)\,\text{Max of } \left(\frac{e^z}{z+2i}\right) \text{ on } C$

length of $C = 6 \cdot 4 = 24$

$|z+2i| \geq |z| - 2 \geq 3 - 2 = 1$

$|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = e^x \leq e^3$

So $|I| \leq 24 \cdot e^3$

(b) $I = \int_C \frac{e^z}{z+2i} dz = 2\pi i \, f(-2i)$

$= 2\pi i \, e^{-2i}$

by the Cauchy integral formula