(1) (15 points) Using the definition of $E,F,G,l,m,n$, prove the matrix identity

$$
\begin{pmatrix}
  l & m \\
  m & n
\end{pmatrix} =
\begin{pmatrix}
  E & F \\
  F & G
\end{pmatrix}
\begin{pmatrix}
  a & c \\
  b & d
\end{pmatrix}
$$

where

$$
\begin{pmatrix}
  a & c \\
  b & d
\end{pmatrix}
$$

is the matrix of the shape operator in the basis $\varphi_u, \varphi_v$.

Solution: Write $S_p(\varphi_u) = a\varphi_u + b\varphi_v, S_p(\varphi_v) = c\varphi_u + d\varphi_v$.

$l = S(\varphi_u)\cdot\varphi_u = aE + bF, \quad m = S(\varphi_u)\cdot\varphi_v = aF + bG = S(\varphi_v)\cdot\varphi_u = cE + dF, \quad n = S(\varphi_v)\cdot\varphi_v = cF + dG$.

(2) (26 points) At a certain point $p$ on a certain parametrized surface $M$, we find $E = 4, F = 1, G = 1, l = 6, m = 3, n = 3$. At this point,

(a) find the matrix of the shape operator with respect to the basis $\varphi_u, \varphi_v$,

Solution: Given $S_p(\varphi_u) = 4\varphi_u + \varphi_v, S_p(\varphi_v) = \varphi_u + 3\varphi_v$,

$$
\begin{pmatrix}
  l & m \\
  m & n
\end{pmatrix} =
\begin{pmatrix}
  E & F \\
  F & G
\end{pmatrix}
\begin{pmatrix}
  a & c \\
  b & d
\end{pmatrix} =
\begin{pmatrix}
  \frac{1}{3} & 1 & -1 & 6 & 3 \\
  -1 & 4 & 3 & 3
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  2 & 3
\end{pmatrix}.
$$

(b) find the principal curvatures $k_1, k_2$,

Solution: the determinant of $S - \lambda I d$ is $(1 - \lambda)(3 - \lambda)$. The roots of this: 1 and 3, are the principal curvatures.

(c) show that the principal directions are $u_1 = \varphi_v, u_2 = \frac{\varphi_u - \varphi_v}{\sqrt{3}}$. What is the angle between $u_1$ and $u_2$?

Solution: By the above, $S(\varphi_v) = 3\varphi_v$. So $u_1 = \varphi_v$ because $\varphi_v$ has length 1 because $G = 1$. The principal directions are always orthogonal because $S$ is symmetric. We have

$$
S(\varphi_u - \varphi_v) = \varphi_u - \varphi_v
$$
So $\varphi_u - \varphi_v$ is an eigenvector. 

$$|\varphi_u - \varphi_v| = \sqrt{(\varphi_u - \varphi_v)(\varphi_u - \varphi_v)} = \sqrt{E - 2F + G} = \sqrt{3}$$

so $u_2 = \frac{\varphi_u - \varphi_v}{\sqrt{3}}$.

(3) (15 points) Let $\varphi(u, v)$ be a patch for a surface $M$ defined on a domain $D \subset \mathbb{R}^2$. Assume that, for all $(u,v) \in D$, the point $p = \varphi(u,v)$ is umbilic on the surface $M$ with constant nonzero principal curvature $k$. Prove that the patch is contained in a sphere.

**Solution:** The image of $\varphi$ “should” be contained in a sphere of radius $\frac{1}{k}$ and center $\varphi + \frac{1}{k}U$. So, as in the case of evolutes of plane curves, we take the derivatives of $\varphi + \frac{1}{k}U$ and show that they are 0 to show that $\varphi + \frac{1}{k}U$ is constant.

$$\frac{\partial}{\partial u} \left( \varphi + \frac{1}{k}U \right) = \varphi_u + \frac{1}{k}U_u = \varphi_u - \frac{1}{k}S_p(\varphi_u) = 0 = \frac{\partial}{\partial v} \left( \varphi + \frac{1}{k}U \right).$$

So $p := \varphi + \frac{1}{k}U$ is constant and

$$|\varphi - p| = \frac{1}{k}$$

which shows that the patch lies on the sphere of center $p$ and radius $\frac{1}{k}$.

(4) (44 points) Let $M$ be the torus obtained by rotating the circle of radius 1 and center $(3, 0, 0)$ in the $xz$ plane around the $z$-axis.

(a) Write a patch $\varphi(u,v)$ for $M$. What is the domain of $\varphi(u,v)$? **Solution:**

$$\varphi(u, v) = ((3 + \cos u) \cos v, (3 + \cos u) \sin v, \sin u).$$

$u \in ]0, 2\pi[, v \in ]0, 2\pi[.$

(b) Compute $\varphi_u, \varphi_v, U, E, F, G, l, m, n$. **Solution:**

$$\varphi_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \quad \varphi_v = (-3 + \cos u) \sin v, (3 + \cos u) \cos v, 0).$$

$$\varphi_u \times \varphi_v = \begin{vmatrix} i & j & k \\ -\sin u \cos v & -\sin u \sin v & \cos u \\ -(3 + \cos u) \sin v & (3 + \cos u) \cos v & 0 \end{vmatrix} = -\begin{vmatrix} 0 & (3 + \cos u) \cos v & (3 + \cos u) \sin v \end{vmatrix}$$

$$U = (\cos v \cos u, \sin v \cos u, \sin u)$$

$$\varphi_{uu} = (-\cos u \cos v, -\cos u \sin v, -\sin u), \quad \varphi_{uv} = (\sin u \sin v, -\sin u \cos v, 0),$$

$$\varphi_{vv} = (-3 + \cos u) \cos v, -(3 + \cos u) \sin v, 0)$$

$$E = \varphi_u \cdot \varphi_u = 1, \quad F = \varphi_u \cdot \varphi_v = 0, \quad G = \varphi_v \cdot \varphi_v = 3 + \cos u$$
\[ l = \varphi_{uu} \cdot U = -1, \quad m = \varphi_{uv} \cdot U = 0, \quad n = \varphi_{vv} \cdot U = -(3 + \cos u) \cos u \]

(c) Compute the Gaussian curvature \( K \) of \( M \) at an arbitrary point \( p = \varphi(u, v) \). **Solution:**
\[ K = \frac{ln - m^2}{EG - F^2} = \cos u \]

(d) Identify the points where \( K \) is positive, zero or negative. Explain where these points lie on the torus.

**Solution:** \( K \) is 0 if \( \cos u = 0 \) which means \( u = \pi/2, 3\pi/2 \). These are the points on top most horizontal circle and the bottom most horizontal circle on the torus. \( K < 0 \) if \( \cos u < 0 \) which means \( u \in ]\pi/2, 3\pi/2[ \). These are the points between the two above circles in the interior surface of the torus. \( K > 0 \) if \( \cos u > 0 \) which means \( u \in ]0, \pi/2[ \cup ]3\pi/2, 2\pi[ \). These are the points between the two above circles in the exterior surface of the torus.