Brief return to the hyperboloid with one sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

find one line from each ruling, then rotate it:

$x = at \quad y = b \quad z = ct$

line: $\alpha(t) = (at, b, ct)$
$= (0, b, 0) + t(a, 0, c)$

directrix: $\beta(u) = (a\cos u, b\sin u, 0)$

a general line of the ruling:

$\alpha_u(t) = (a\cos u, b\sin u, 0)$
$+ t(a\sin u, b\cos u, c)$

e.g. change $c \to -c$ to get the second ruling.

<u>Back to geodesics</u>: M surface.

$\varphi(u, v)$ a patch for M.

curve α on M: $\alpha(t) = \varphi(u(t), v(t))$

$\alpha'(t)$ velocity

$\alpha''(t)$ acceleration

α is a geodesic when α'' is normal or perpendicular to M.

$$\alpha'(t) = \varphi_u \cdot u'(t) + \varphi_v \cdot v'(t)$$

$$\alpha''(t) = u''(t)\,\varphi_u + v''(t)\,\varphi_v$$
$$u'(t)^2\varphi_{uu} + u'(t)v'(t)\,\varphi_{uv}$$
$$+ u'(t)v'(t)\,\varphi_{uv} + v'(t)^2\,\varphi_{vv}$$

bases for $\mathbb{R}^3$: $T,\ U \times T,\ U$

or $\varphi_u,\ \varphi_v,\ U$

<u>Recall</u> the Christoffel symbols:

$$\varphi_{uu} = \Gamma^u_{uu}\varphi_u + \Gamma^v_{uu}\varphi_v + l\,U$$
$$\varphi_{uv} = \Gamma^u_{uv}\varphi_u + \Gamma^v_{uv}\varphi_v + m\,U$$
$$\varphi_{vv} = \Gamma^u_{vv}\varphi_u + \Gamma^v_{vv}\varphi_v + n\,U$$

<u>Recall</u>: the first fundamental form: or metric

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$E = \varphi_u \cdot \varphi_u \quad F = \varphi_u \cdot \varphi_v \quad G = \varphi_v \cdot \varphi_v$

computes lengths of tangent vectors on M

the second fundamental form:

$$\begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

$$l = S(\varphi_u) \cdot \varphi_u = \varphi_{uu} \cdot U$$

$$m = S(\varphi_u) \cdot \varphi_v = S(\varphi_v) \cdot \varphi_u = \varphi_{uv} \cdot U$$

$$n = S(\varphi_v) \cdot \varphi_v = \varphi_{vv} \cdot U.$$

$v, w \in T_p M$

$$v = \lambda_1 \varphi_u + \mu_1 \varphi_v$$

$$w = \lambda_2 \varphi_u + \mu_2 \varphi_v.$$

$$v \cdot w = (\lambda_1 \ \ \mu_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}$$

symmetric bilinear form.

= first fundamental form.

$v, w \in T_p M$

$$\mathrm{II}(v, w) = S(v) \cdot w.$$

symmetric bilinear form = second fundamental form

$$= \begin{pmatrix} \lambda_1 & \mu_1 \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}$$

Last quarter we expressed the Christoffel symbols in terms of the first ~~and second~~ fundamental forms:

$$\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_v \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^u_{vv} \\ \Gamma^v_{vv} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$\alpha'' = \left(u'' + u'^2\Gamma^u_{uu} + 2u'v'\Gamma^u_{uv} + v'^2\Gamma^u_{vv}\right)\varphi_u$$
$$+ \left(v'' + v'^2\Gamma^v_{vv} + 2u'v'\Gamma^v_{uv} + u'^2\Gamma^v_{uu}\right)\varphi_v$$
$$+ \left(u'^2 l + 2u'v' m + v'^2 n\right) U$$

α is a geodesic $\Longleftrightarrow$

$$\begin{cases} u'' + u'^2\Gamma^u_{uu} + 2u'v'\Gamma^u_{uv} + v'^2\Gamma^u_{vv} = 0 \\ \& \ v'' + v'^2\Gamma^v_{vv} + 2u'v'\Gamma^v_{uv} + u'^2\Gamma^v_{uu} = 0 \end{cases}$$

$\hookrightarrow$ These are the geodesic equations.

$$\Gamma^u_{uu} = \frac{\frac{1}{2}GE_u - F\left(F_u - \frac{1}{2}E_v\right)}{EG - F^2}$$

$$\Gamma^v_{uu} = \frac{-\frac{1}{2}FE_u + E\left(F_u - \frac{1}{2}E_v\right)}{EG - F^2}$$

$$\Gamma^u_{uv} = \frac{1}{2}\,\frac{GE_v - FG_u}{EG - F^2}$$

$$\Gamma^v_{uv} = \frac{1}{2}\,\frac{-FE_v + EG_u}{EG - F^2}$$

$$\Gamma^u_{vv} = \frac{-\frac{1}{2}FG_v + G\left(F_v - \frac{1}{2}G_u\right)}{EG-F^2}$$

$$\Gamma^v_{vv} = \frac{\frac{1}{2}EG_v - F\left(F_v - \frac{1}{2}G_u\right)}{EG-F^2}$$

First geodesic equation in terms of the metric: (multiply through by $EG-F^2$)

$$(EG-F^2)u'' + u'^2\left(\frac{1}{2}GE_u + \frac{1}{2}FE_v - FF_u\right)$$

$$+ 2u'v'\left(\frac{1}{2}(GE_v - FG_v)\right)$$

$$+ v'^2\left(-\frac{1}{2}FG_v - \frac{1}{2}GG_u + GF_v\right) = 0$$

switch u and v to get the second equation.

Theorem 5.2.3: M surface given by patch $\varphi(u,v)$. ~~$\alpha(t) = \varphi(u$~~ given $P \in M$ and $V \in T_P M$, $\exists!$ geodesic $\alpha(t)$ on M

such that $\alpha(0) = P$ and $\alpha'(0) = V$.

Proof: Follows from the existence and uniqueness of solutions to the geodesic equations which are a system of two differential equations of second order. $\Box$.

Examples: geodesics on the sphere and the torus. (Section 5.2).

Clairaut geodesic equations:

u- Clairaut: $F=0$, $E_v = G_v = 0$

$$EGu'' + \frac{1}{2} u'^2 GE_u - \frac{1}{2} v'^2 GG_u = 0$$

$$EGv'' + u'v' EG_u = 0$$

v- Clairaut: $F=0$, $E_u = G_u = 0$

$$EGv'' + \frac{1}{2} v'^2 EG_v - \frac{1}{2} u'^2 EE_v = 0$$

$$EGu'' + u'v' GE_v = 0$$

Theorem: 5.2.7: If we have a

u-Clairaut coordinate chart, then the u-parameter curves are geodesics.

<u>Example:</u> <u>torus:</u>

$$\varphi(u,v) = \big((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\big)$$

$$\varphi_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$

$$\varphi_v = (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0)$$

$$\varphi_u \cdot \varphi_v = 0 = F$$

$$\varphi_u \cdot \varphi_u = r^2 = E \quad \text{constant.}$$

$$\varphi_v \cdot \varphi_v = (R + r\cos u)^2 = G.$$

$$G_v = 0.$$

u-Clairaut $\Rightarrow$ u-parameter curves are geodesis.

$$\alpha_{v_0}(u) = \varphi(u, v_0) = \big((R + r\cos u)\cos v_0, (R + r\cos u)\sin v_0, r\sin u\big)$$