Theorem (5.2.8) Suppose M has a u-Clairaut parametrization $\varphi(u,v)$. (1) Then the u-parameter curves are geodesics (after reparametrization to have constant speed) (2) The v-parameter curves (which ~~are~~ is automatically constant speed) ~~are~~ is a geodesic at u_0 if and only if $G_u(u_0)=0$

Proof: (1) $\alpha(u)=\varphi(u,v_0)$

$$s(u)=\int_0^u |\alpha'(\gamma)|\,d\gamma$$

$$|\alpha'(\gamma)|=\sqrt{\alpha'(\gamma)\cdot\alpha'(\gamma)}=\sqrt{\varphi_u\cdot\varphi_u}=\sqrt{E(\gamma)}$$

$$s(u)=\int_0^u\sqrt{E(\gamma)}\,d\gamma$$

So $\dfrac{ds}{du}=\sqrt{E(u)}>0$

So s is a strictly increasing function of u, so s is 1-to-1, So $s(u)$ has an inverse function

$u(s)$.

Define $\beta(s) = \alpha(u(s)) = \varphi(u(s), v_0)$

$$\beta'(s) = \frac{d\alpha}{du}\frac{du}{ds} = \varphi_u \cdot \frac{1}{\frac{ds}{du}}$$

$$= \varphi_u \cdot \frac{1}{\sqrt{E}} = \frac{\alpha'}{|\alpha'|}$$

$$|\beta'(s)| = |\varphi_u| \cdot \frac{1}{\sqrt{E}} = 1$$

To show that β is a geodesic, we show that it satisfies the geodesic equations.

$$E u'' + \frac{1}{2} u'^2 E_u - \frac{1}{2} v'^2 G_u = 0$$

$$G v'' + u' v' G_u = 0$$

The second equation is satisfied because $v' = v'' = 0$. The first equation simplifies to:

$$E u'' + \frac{1}{2} u'^2 E_u = 0.$$

<u>recall</u>: $u' = \frac{du}{ds} = \frac{1}{\frac{ds}{du}} = \frac{1}{\sqrt{E(u)}}$

$$u'' = \frac{d}{ds}\left(\frac{du}{ds}\right) = \frac{d}{ds}\left(\frac{1}{\sqrt{E(u)}}\right)$$

$$= \frac{d}{du}\left(\frac{1}{\sqrt{E(u)}}\right) \cdot \frac{du}{ds}$$

$$= -\frac{1}{2} E^{-3/2} E_u \frac{1}{\sqrt{E}} = -\frac{E_u}{2E^2}.$$

So $E u'' + \frac{1}{2} u'^2 E_u =$

$$E \cdot \left(-\frac{E_u}{2E^2}\right) + \frac{1}{2}\frac{1}{E} \cdot E_u = 0$$

(2) $\alpha(v) = \varphi(u_0, v)$ $\quad \alpha' = \varphi_v.$

$|\alpha'| = \sqrt{\varphi_v \cdot \varphi_v} = \sqrt{G}$ independent of v

So $|\alpha'|$ is constant if u is constant.

The Clairaut equations become:

$$v'^2 G_u = 0 \text{ and } G v'' = 0$$

$$\Leftrightarrow v'' = 0.$$

$\cancel{v' \neq 0}$ $\Leftrightarrow G_u = 0$

$\qquad\quad \overset{\shortparallel}{G_u(u_0)}$

v is the parameter, so $v' = 1$, $v'' = 0$

We have a geodesic iff $G_u(u_0) = 0$ $\quad \square$

Example: A surface of revolution:

$$\varphi(u,v) = (g(u), h(u)\cos v, h(u)\sin v).$$

$$\gamma(u) = (g(u), h(u)) \text{ in the } xy \text{ plane.}$$

$E = \varphi_u \cdot \varphi_u$ $\quad \varphi_u = (g', h'\cos v, h'\sin v)$

$E = g'^2 + h'^2$ $\quad$ so $E_v = 0$

$\varphi_v = (0, -h\sin v, h\cos v)$

$F = \varphi_u \cdot \varphi_v = 0$

$G = \varphi_v \cdot \varphi_v = h^2$ $\quad$ so $G_v = 0$

$\varphi(u, v_0)$ is a meridian.
(a rotation of γ)

The meridians are geodesics

$\varphi(u_0, v)$ is a parallel (a circle)

it is a geodesic when $G_u(u_0) = 0$.

i.e., $hh'(u_0) = 0$

$h \neq 0$ because $h^2 = G = \varphi_v \cdot \varphi_v \neq 0$

So a parallel is a geodesic exactly when $h'(u_0) = 0$.

By Exercise 5.2.2, the two geodesic equations imply the constant speed equation and one of the geodesic equations with the constant speed equation imply the other geodesic equation.

Assume again that we have a u-Clairaut parametrization. The second u-Clairaut equations can be written as:

$$\frac{v''}{v'} = -u'\frac{G_u}{G}$$

$$\int \frac{v''}{v'}\,dt = -\int u'\frac{G_u}{G}\,dt$$

$$\ln v' = -\ln G + \text{constant}$$

$$\Rightarrow \quad v' = -\frac{c}{G}$$

Now look at the unit speed equation

$$1 = |\alpha'(t)| \qquad \alpha = \varphi(u(t), v(t))$$

$$\alpha' = u'\varphi_u + v'\varphi_v$$

$$\alpha' \cdot \alpha' = u'^2 \varphi_u \cdot \varphi_u + 2u'v' \varphi_u \cdot \varphi_v + v'^2 \varphi_v \cdot \varphi_v$$

$$= E u'^2 + G v'^2$$

So $1 = E u'^2 + G v'^2$

substitute in $v' = -\dfrac{c}{G}$

$$1 = E u'^2 + \frac{c^2}{G}$$

$$\Rightarrow \quad u'^2 = \frac{1}{E} - \frac{c^2}{EG}$$

$$u' = \pm \sqrt{\frac{G - c^2}{EG}}$$

$$\frac{dv}{du} = \frac{v'}{u'} = \pm \frac{\frac{c}{G}}{\sqrt{\frac{G-c^2}{EG}}}$$

$$\frac{dv}{du} = \frac{\pm c\sqrt{E}}{\sqrt{G}\sqrt{G-c^2}}$$

$$\Rightarrow \boxed{v = \pm \int \frac{c\sqrt{E}}{\sqrt{G(G-c^2)}}\, du}$$

This determines all geodesics.

Completeness:

Definition: A surface M is called complete if the domain of every geodesic is $\mathbb{R}$.

Example: $\mathbb{R}^2 - \{0\} = M$.

line $t(a, b)$ is a geodesic

the domain is $\mathbb{R} \setminus \{0\}$

Geodesics are locally the shortest paths on a surface.

M a surface. $P \in M$

$\exists$ coordinate chart for M near P s.t. $E = 1$, $F = 0$, $G > 0$ (geodesic polar coordinate chart).

$\varphi(u, v) = \alpha_w(u)$

where $w = \cos v\, e_1 + \sin v\, e_2$

(e_1, e_2) basis of T_pM orthonormal.