Differentials of maps of surfaces:

$F : M \longrightarrow N$ map of surfaces.

want to define maps

$$F_* : T_p M \longrightarrow T_{F(p)} N \quad p \in M.$$

which is ~~are~~ the differential of F at p.
exists when the map is differentiable

Def: $F : M \longrightarrow N$ is differentiable if for all coordinate charts φ of M, ψ of N, the map

$$\psi^{-1} F \varphi : D_\varphi \longrightarrow D_\psi \subset \mathbb{R}^2.$$

is differentiable.

Def: Suppose $F : M \to N$ is differentiable, then $\forall\, v \in T_p M$, choose α curve in M s.t. $\alpha(0) = p$ and $\alpha'(0) = v$, then $F(\alpha(t)) = \beta(t)$ ~~$F_*(v) =$~~ is a curve in N.

$$F_*(v) := \beta'(0) = \frac{d}{dt}\left(F(\alpha(t))\right)\Big|_{t=0}.$$

<u>**Example:**</u> $M = N = \mathbb{R}^2$.

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (f(x,y), g(x,y))$$

$\alpha(t) = (x(t), y(t))$ $\qquad \alpha(0) = (x_0, y_0)$

$\qquad \alpha'(0) = (\lambda, \mu).$

$$\beta(t) = F(\alpha(t)) = \left(f(x(t), y(t)), g(x(t), y(t))\right)$$

$$\beta'(0) = \Big(\frac{\partial f}{\partial x}(x_0,y_0)\cdot x'(0) + \frac{\partial f}{\partial y}(x_0,y_0)\, y'(0),$$

$$\frac{\partial g}{\partial x}(x_0,y_0)\, x'(0) + \frac{\partial g}{\partial y}(x_0,y_0)\, y'(0)\Big)$$

$$= \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}}_{\text{The Jacobian matrix of } F.} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$

You will see in homework that F_* is

linear, so it's determined by $F_*(\varphi_u)$ and $F_*(\varphi_v)$

T_pM has basis φ_u, φ_v

$T_{F(p)}N$ has basis ψ_x, ψ_y.

If $v = \lambda\varphi_u + \mu\varphi_v$

and if $F_*(\varphi_u) = a\psi_x + b\psi_y$

$F_*(\varphi_v) = c\psi_x + d\psi_y$,

then $F_*(v) = \lambda F_*(\varphi_u) + \mu F_*(\varphi_v)$

$= \lambda(a\psi_x + b\psi_y) + \mu(c\psi_x + d\psi_y)$

$= (\lambda a + \mu c)\psi_x + (\lambda b + \mu d)\psi_y$

So in the bases $\{\varphi_u, \varphi_v\}$ of T_pM

and $\{\psi_x, \psi_y\}$ of $T_{F(p)}N$,

the matrix of F_* is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

<u>Example:</u> Suppose M is an oriented surface. The Gauss map

$$G : M \longrightarrow S^2 \subset \mathbb{R}^3.$$

$$p \longmapsto U(p) \quad \text{unit normal at } p.$$

$p \in M$. $\quad G_* : T_pM \longrightarrow T_{G(p)} S^2$.

$$\begin{array}{ccc} T_pM & & T_{G(p)}S^2 \\ \| & & \| \\ \text{plane} \perp U(p) & & \text{plane} \perp U(p) \\ & & \overset{\|}{G(p)} \end{array}$$

So we can identify T_pM with $T_{G(p)}S^2$ and think of

$$G_* : T_pM \longrightarrow T_pM$$

$v \in T_pM$. $\quad \alpha$ a curve in M s.t.

$$\alpha(0) = p \quad \alpha'(0) = v$$

$$G_*(v) = G_*(\alpha'(0)) = \frac{d}{dt} G(\alpha(t)) \Big|_{t=0}$$

$$= \frac{d}{dt} U(\alpha(t)) \Big|_{t=0}$$

$$= \nabla_{\alpha'(0)} U = \nabla_v U$$

$$= - S_p(v)$$

<u>Def</u>: (1) We say two surfaces are diffeomorphic if $\exists$ $F: M \to N$ differentiable s.t. F has an inverse map $F^{-1}: N \to M$ which is also differentiable.

(2) We say M and N are locally diffeomorphic at $p \in M, q \in N$ if $\exists$ small open $p \in U \subset M$ and small open $q \in V \subset N$ s.t. U & V are diffeomorphic via a map sending p to q.

<u>The inverse function theorem:</u>

<u>Recall</u>: $f: \mathbb{R} \to \mathbb{R}$, if $f'(a) \neq 0$ then $\exists$ small open interval $(a-\varepsilon, a+\varepsilon)$ s.t. $\forall x \in$ $(a-\varepsilon, a+\varepsilon)$ $f'(x) \neq 0$ provided f is C^1

So on $(a-\varepsilon, a+\varepsilon)$ $f'(x)$ is always either positive or negative.

if $f'(x) > 0$ $\forall x$, then f is strictly increasing, hence 1-to-1, hence f has an inverse, f^{-1} is also C^1.

if $f'(x) < 0$ $\forall x$, then f is strictly decreasing, hence 1-to-1, hence f has an inverse f^{-1} which is C^1

<u>Generalization to surfaces:</u>

$F: M \longrightarrow N$ ~~of~~ C^1

suppose $p \in M$ s.t. F_{*p} is invertible as a linear map $T_pM \longrightarrow T_{F(p)}N$

(can take <u>$\det F_*$</u>, this should be $\neq 0$)

then $\exists$ small neighborhood U of $p \in M$ and a small neighborhood V of $F(p) \in N$ s.t. F has an inverse $F^{-1}: V \longrightarrow U$. The inverse is also C^1.

The exponential map:

$p \in M \qquad T_pM.$

the exponential map will send a small neighborhood of $0 \in T_pM$ to a small neighborhood of $p \in M$

diffeomorphically.

$v \in T_pM. \quad \exists$ geodesic γ_v s.t. $\gamma_v(t)$

$\gamma_v(0) = p$ and $\gamma_v'(0) = v$

Suppose v has small enough length so that γ_v is defined on $[-1, 1]$.

Def: $\exp_p(v) := \gamma_v(1)$

Lemma: $\exp_p(tv) = \gamma_v(t)$.

Proof: $\exp_p(tv)$ is $\gamma_{tv}(1)$

γ_{tv} is the geodesic through p with velocity tv at 0. t fixed.

$\underline{\gamma_v(s)} \qquad \underline{\gamma_v(ts)}$

$\gamma_v(ts)$ takes the value p at 0.

$$(\gamma_v(ts))' = \frac{d}{ds}(\gamma_v(ts)) = t\,\gamma'(ts).$$

so at 0 $\quad (\gamma_v(ts))'\Big|_{s=0} = t\,\gamma'(0) = tv$

uniqueness of geodesics

$$\Rightarrow \gamma_v(ts) = \gamma_{tv}(s) \quad \forall s$$

$$\Rightarrow \gamma_v(t) = \gamma_{tv}(1) = \exp_p(tv) \quad \square$$

geodesic polar coordinates:

$p \in M$. choose a basis e_1, e_2 of $T_pM \cong \mathbb{R}^2$

u = parameter on lines through the origin of T_pM

v = angle of a line with e_1.

(u, v) = polar coordinates on T_pM

$$\cancel{\rho(u,v) =}\ (x,y) = (u\cos v, u\sin v)$$

$$= u\cos v\, e_1 + u\sin v\, e_2$$

$$\rho(u,v) = \gamma_{\cos v\, e_1 + \sin v\, e_2}(u)$$

$$= \gamma_{u\cos v\, e_1 + u\sin v\, e_2}(1)$$

or $\rho(u,v) = \exp_p(u\cos v e_1 + u\sin v e_2)$

$$\rho_u = \frac{\partial}{\partial u} \gamma_{(u\cos v,\, u\sin v)}(1).$$

$$= \frac{\partial}{\partial u} \gamma_{(\cos v, \sin v)}(u).$$

$\rho_u \big|_{u=0}$ = velocity vector of $\gamma_{(\cos v, \sin v)}$ at 0

$= \cos v e_1 + \sin v e_2$.

E at p. $= \rho_u \cdot \rho_u \big|_{u=0} = 1.$