~~Lemma:~~ <u>Recall</u>: $\exp_p : T_pM \longrightarrow M$ from a neighborhood of $0 \in T_pM$ to a neighborhood of $p \in M$.

$0 \in T_pM \cong \mathbb{R}^2$ is a surface.

$$T_0(T_pM) = T_pM$$

because $T_0\mathbb{R}^2 = \mathbb{R}^2$

The differential of $\exp$:

$$\exp_* : T_0(T_pM) = T_pM \longrightarrow T_pM.$$

<u>Lemma</u>: $\exp_*$ is the identity.

<u>Proof</u>: $v \in T_pM$ $\quad \exp_*(v) = v$?

$\alpha(t) := tv \qquad \alpha(0) = 0 \qquad \alpha'(0) = v$

$$\begin{aligned}
\exp_*(v) &= \frac{d}{dt}\left(\exp(\alpha(t))\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\exp(tv)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\gamma_{tv}(1)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\gamma_v(t)\right)\Big|_{t=0} = \gamma'(0) \\
&= v \qquad \square
\end{aligned}$$

Corollary: $\exists \quad \varepsilon > 0$ s.t.

$$\exp\Big|_{D_\varepsilon} : D_\varepsilon \longrightarrow \mathcal{N}_\varepsilon \quad \text{is a}$$

diffeomorphism, where

$$D_\varepsilon = \{ v \mid |v| < \varepsilon \} \subset T_pM = \mathbb{R}^2$$

and $\mathcal{N}_\varepsilon$ is a neighborhood of p in M

Proof: Apply the inverse function theorem, using the lemma. $\square$

Definition: $\mathcal{N}_\varepsilon$ as above is called a normal neighborhood of p.

Back to geodesics:

We use the exponential map to "transfer" the polar coordinates on the plane to the surface.

$$(u, v) = (r, \theta).$$

u-parameter curves are unit speed geodesics
v-parameter curves are images of circles, called polar circles.

Choose a basis e_1, e_2 of T_pM

$x = r\cos\theta = u\cos v$

$y = r\sin\theta = u\sin v.$

vectors in T_pM will be of the form $u\cos v\, e_1 + u\sin v\, e_2$.

map. $\varphi : T_pM \longrightarrow M$

$$\varphi(u,v) = \exp(u\cos v e_1 + u\sin v e_2)$$
$$= \gamma_{u\cos v e_1 + u\sin v e_2}(1)$$
$$= \gamma_{\cos v e_1 + \sin v e_2}(u).$$

$\gamma_{\cos v e_1 + \sin v e_2}$ is a unit speed geodesic because $\gamma'(0) = \cos v e_1 + \sin v e_2$ has length $1 \Rightarrow |\gamma'| = 1$ everywhere because geodesics have constant speed.

$\gamma' = \varphi_u$ So $\gamma' \cdot \gamma' = 1$

$$\Rightarrow \varphi_u \cdot \varphi_u = 1$$
$$\overset{\|}{E} = 1 \text{ everywhere..}$$

$\varphi : T_pM \longrightarrow M$ defined on a neighborhood D_ε of 0.

$\exists$ largest $\varepsilon_1 > 0$ (possibly ∞) s.t. φ is well-defined on D_{ε_1}.

$$\varphi : D_\varepsilon \longrightarrow M$$

is the <u>geodesic polar map</u>.

$\exists$ largest $\varepsilon_0 > 0$ (possibly ∞) s.t. φ is a diffeomorphism from D_{ε_0} to its image $\mathcal{N}_{\varepsilon_0} \subset M$.

$$\varphi : D_{\varepsilon_0} \longrightarrow \mathcal{N}_{\varepsilon_0}$$

is a <u>geodesic polar parametrization</u> with <u>pole p</u>.

$$\varphi_v(0) = 0 \quad \text{because}$$

$$\varphi(0, v) = p \quad \forall v.$$

<u>Lemma:</u> For a geodesic polar parametrization: $E = 1$, $F = 0$, $G > 0$ everywhere except that $G = 0$ at 0.

Proof: We already know that the u-parameter curves are unit speed geodesics so that $E = \varphi_u \cdot \varphi_u = 1$ and φ_{uu} is normal to M.

$\Rightarrow$ $\varphi_{uu} \cdot \varphi_u = \varphi_{uu} \cdot \varphi_v = 0$.

$$F_u = (\varphi_u \cdot \varphi_v)_u = \varphi_{uu} \cdot \varphi_v + \varphi_u \cdot \varphi_{vu}$$

$$= 0 + \varphi_u \cdot \varphi_{uv} = 0$$

because $\underset{\varphi_u \cdot \varphi_{uv}}{\underset{\shortparallel}{E_v}} = E_u = 0$

So F is independent of u

$\Rightarrow$ $F(u,v) = F(0,v) \quad \forall v$

$$\| \quad (\varphi_u \cdot \varphi_v)(0) = 0$$

because $\varphi_v(0) = 0$.

$\Rightarrow$ $F = 0$ everywhere.

Now G: $G = \varphi_v \cdot \varphi_v$

φ is regular on D_ε except at 0.

$\Rightarrow$ $\varphi_v \neq 0$ $\quad G = |\varphi_v|^2 > 0$ except at 0. $\square$.

Claim: Suppose $\varphi_\varepsilon : D_\varepsilon \to \mathcal{N}_\varepsilon \subset M$ is a geodesic polar parametrization with pole p. Then $\forall\ q \in \mathcal{N}_\varepsilon$, geodesics from p to q are the curves of shortest length from p to q in $\mathcal{N}_\varepsilon$.

Proof: Suppose $\alpha : [0, u_0] \to \mathcal{N}_\varepsilon$ is a unit speed geodesic s.t. $\alpha(0) = p$ and $\alpha(u_0) = q = \cancel{f(u_0,?)}\ \varphi(u_0, ?)$

Suppose $\beta : [0, u_0] \to \mathcal{N}_\varepsilon$ is another curve s.t. $\beta(0) = p$ and $\beta(u_0) = q$.

(reparametrize β if necessary to ensure that α & β are defined on the same interval)

α is a u-parameter curve for a geodesic polar parametrization.

The length of α is:

$$L(\alpha) = \int_0^{u_0} \underbrace{|\alpha'|}_{1}\, du = \int_0^{u_0} du = u_0$$

$$L(\beta) = \int_0^{u_0} |\beta'|\, du$$ ~~write $\beta(u) = f(u)$~~

write $\beta(u) = \varphi(f(u), g(u))$

$$\beta' = f'\varphi_u + g'\varphi_v$$

$$|\beta'| = \sqrt{Ef'^2 + 2Ff'g' + Gg'^2}$$

$$= \sqrt{f'^2 + G\,g'^2}$$

$$L(\beta) = \int_0^{u_0} \sqrt{f'^2 + Gg'^2}\,du$$

$$\geqslant \int_0^{u_0} f'\,du = f(u_0) - f(0)$$

$$= f(u_0) = u_0 = L(\alpha)$$

$\square$

Remark: Since $\exists !$ line in plane through any two points, $\exists !$ geodesic from p to any point $q \in \mathcal{V}_\varepsilon$ normal neighborhood

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Back to completeness:

Theorem (Hopf-Rinow, 5.3.1) If M is geodesically complete, then any two points can be joined by a geodesic and curves of shortest length between two points are geodesics.

<u>Definition:</u> A surface $M \subset \mathbb{R}^3$ is closed if $\forall$ sequence $\{z_n\}$ of points of M s.t. $\lim_{n\to\infty} z_n = z \in \mathbb{R}^3$

we have $z \in M$.

<u>Theorem:</u> If $M \subset \mathbb{R}^3$ is closed, then M is geodesically complete.

<u>Surfaces NOT in $\mathbb{R}^3$:</u>

How do we measure length on M?

We need to measure lengths of velocity vectors of curves, i.e., lengths of tangent vectors.

<u>Def</u>: An abstract surface is the union of images of domains in $\mathbb{R}^2$.

<u>e.g.</u>: $D_1 \subset \mathbb{R}^2 \supset D_2$
open discs in $\mathbb{R}^2$

choose $\underset{\text{open}}{U_1 \subset D_1}$ $\underset{\text{open}}{U_2 \subset D_2}$

and a diffeomorphism $\varphi : U_1 \twoheadrightarrow U_2$

$D_1 \cup D_2$ identify $p \in U_1$ with $\varphi(p) \in U_2$.